Probabilistic Engineering Mechanics 28 (2012) 152–163

Contents lists available at SciVerse ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Stochastic analysis of structures with uncertain-but-bounded parameters via improved interval analysis

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ARTICLE INFO

Article history:
Received 11 June 2011
Received in revised form
19 July 2011
Accepted 17 August 2011
Available online 27 August 2011

Keywords:
Uncertain-but-bounded parameters
Random excitation
Interval analysis
Dependency phenomenon
Affine arithmetic
Stochastic analysis
Covariance vector
Upper and lower bounds

ABSTRACT

The stochastic analysis of linear structures, with slight variations of the structural parameters, subjected to zero-mean Gaussian random excitations is addressed. To this aim, the fluctuating properties, represented as uncertain-but-bounded parameters, are modeled via *interval analysis*. In the paper, a novel procedure for estimating the lower and upper bounds of the second-order statistics of the response is proposed. The key idea of the method is to adopt a first-order approximation of the random response derived by properly improving the ordinary interval analysis, based on the philosophy of the so-called *affine arithmetic*. Specifically, the random response is split as sum of two aliquots: the midpoint or nominal solution and a deviation. The latter is approximated by superimposing the responses obtained considering one uncertain-but-bounded parameter at a time. After some algebra, the sets of first-order ordinary differential equations ruling the midpoint covariance vector and the deviations due to the uncertain parameters separately taken are obtained. Once such equations are solved, the region of the response covariance vector is determined by handy formulas.

To validate the procedure, two structures with uncertain stiffness properties under uniformly modulated white noise excitation are analyzed.

1. Introduction

The non-deterministic treatment of external excitations and structural parameters has been the subject of many contributions in the last few years. Indeed, it has been largely recognized that the main excitation sources involved in structural engineering problems, such as earthquake ground motion, wind actions etc., are random in nature and can be modeled, with good accuracy, as Gaussian stochastic processes. In particular, if a linear system is forced by a Gaussian random process, the response is Gaussian too, namely only the statistical moments up to the second-order are needed to fully characterize the stochastic response (see e.g. [1,2]).

Another class of uncertainties occurring in engineering problems is the one associated with fluctuations of structural parameters, such as geometrical and mechanical properties or boundary conditions. These sources of uncertainty, which affect to a certain extent the structural response, are usually described following two contrasting points of view, known as probabilistic and non-probabilistic approaches. The first ones are certainly the most widely adopted and can be developed by three main ways: the Monte Carlo simulation method, the perturbation techniques [3] and the spectral methods [4]. Unfortunately, the probabilistic approaches require a wealth of data, often unavailable, to define the probability density function of the uncertain structural parameters. Furthermore, as stated by the engineering scientist Freudenthal [5], who was one of the pioneers of probabilistic methods in engineering, "…ignorance of the cause of variation does not make such variation random …". This means that, when crucial information on a variability is missing, it is not a good practice to model it as a probabilistic quantity [6]. The recent development of the non-probabilistic approaches stems from the firm belief that the non-probabilistic concepts could be more appropriate to model certain types of non-deterministic information, resulting in a better representation of the simulated non-deterministic physical behavior.

Starting from the pioneering study of Ben-Haim and Elishakoff [7], in the last few years several non-probabilistic approaches have been proposed to perform the static and dynamic analysis of structures. These approaches are mainly based on convex models, interval models and fuzzy sets [8].

Today, the interval model may be considered as the most widely used analytical tool. This model is derived from the *interval analysis* [9–12] in which the number is treated as an interval variable with lower and upper bounds. The main advantage

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doi:10.1016/j.probengmech.2011.08.011

of the interval analysis is that it provides analytically rigorous enclosures of the solution. However, the application of the interval analysis method to real engineering problems is quite difficult. Moreover, this algebra suffers from the so-called *dependency phenomenon*[6,12,13]. This phenomenon frequently occurs in interval analysis when an expression contains multiple instances of one or more interval variables. It follows that the ordinary interval analysis often leads to an overestimation of the interval width that could be catastrophic from an engineering point of view. Indeed, when the operands are partially dependent on each other, not all combinations of values in the given intervals will be valid and the exact interval will probably be smaller than the one produced by the formulas. To limit the catastrophic effects of the dependency phenomenon, the so-called *generalized interval analysis* [14] and *affine arithmetic* [15,16] have been introduced as improvements of the ordinary interval analysis. In these formulations, each intermediate result is represented by a linear function with a small remainder interval [17].

In the case of slight parameter fluctuations and deterministic loads, the so-called *interval perturbation method* (*IPM*) or, equivalently, the *First-Order Interval Taylor Series Expansion* have been successfully adopted in both static [18–22] and dynamic analysis [10,23–25]. The main advantages of these methods are the flexibility and the simplicity of the mathematical formulation. However, since the effect of neglecting the higher-order terms is unpredictable, the effectiveness of these approaches is limited to uncertainties with small intervals.

The aim of this paper is to determine the region of the random response of linear structural systems with uncertain-but-bounded parameters subjected to stochastic excitations modeled as zero-mean Gaussian random processes. To the best of our knowledge, so far such a problem has been tackled only by the authors [26,27] who combined the *IPM* with the differential equations governing the response covariance vector [28].

In this study, an alternative and more general approach, based on a first-order approximation of the response, conceptually different from the one assumed by the *IPM*, is presented. Specifically, the method adopts an improvement of the ordinary interval analysis, based on the philosophy of the so-called affine arithmetic, introducing a particular unitary interval and splitting the random response as sum of two aliquots: the midpoint or nominal solution and the deviation. The latter is approximated by superimposing the responses obtained considering one uncertain-but-bounded parameter at a time. By using Kronecker algebra [29] in conjunction with modal analysis, the differential equations governing the time-evolution of the midpoint and deviation covariance vectors of the response under zero-mean Gaussian stochastic input process are first derived. Once such equations are solved, the upper and lower bounds of the response covariance vector are evaluated by applying handy formulas. It is worth mentioning that the *IPM* recently developed by the authors [27] to deal with stochastic excitations may be viewed as a particular case of the proposed one when appropriate simplifications are introduced.

Numerical results concerning a shear-type frame and a truss structure with uncertain-but-bounded stiffness properties under uniformly modulated white noise excitation are presented to show the effectiveness of the proposed method.

2. The dependency phenomenon in interval analysis

In the context of ordinary interval analysis [9–12], real numbers are bounded by intervals; and basic operations and functions are extended to operate on intervals. More precisely, a real number $x \in \mathbb{R}$ is bounded by an interval $\left[\underline{x}, \bar{x}\right]$, where $\underline{x}$ and $\bar{x}$ are floating-point numbers with understanding that $\underline{x} \le x \le \bar{x}$. Then, $\underline{x}$ and $\bar{x}$ define the lower and upper bounds of the interval, respectively.

Let us now define three closed bounded intervals x^I, y^I and z^I as sets of real numbers, given by

$$x^I \triangleq \left[\underline{x}, \bar{x}\right] = \left\{x \,|\, \underline{x} \le x \le \bar{x},\ x \in \mathbb{R}\right\}; \tag{1a}$$

$$y^I \triangleq \left[\underline{y}, \bar{y}\right] = \left\{y \,|\, \underline{y} \le y \le \bar{y},\ y \in \mathbb{R}\right\}; \tag{1b}$$

$$z^I \triangleq \left[\underline{z}, \bar{z}\right] = \left\{z \,|\, \underline{z} \le z \le \bar{z},\ z \in \mathbb{R}\right\}, \tag{1c}$$

where $\mathbb{R}$ represents the set of all real numbers; the notation $\{\alpha \mid P(\alpha)\}$ means "the set of α such that the proposition $P(\alpha)$ holds". Denoting by $\mathbb{IR}$ the set of all closed real interval numbers, then it is possible to define $x^I, y^I, z^I \in \mathbb{IR}$. An interval is called *thin* if $\underline{x} = \bar{x}$, that is $x^I \triangleq \left[\underline{x}, \underline{x}\right]$, in this case $x \in \mathbb{R}$. For the interval numbers x^I and y^I, the basic algebraic operations are listed below:

$$x^I + y^I = \left[\underline{x} + \underline{y},\ \bar{x} + \bar{y}\right]; \tag{2a}$$

$$x^I - y^I = \left[\underline{x} - \bar{y},\ \bar{x} - \underline{y}\right]; \tag{2b}$$

$$x^I \times y^I = \left[\min\left(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\right), \max\left(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\right)\right]; \tag{2c}$$

$$x^I / y^I = \left[\underline{x},\ \bar{x}\right] \times \left[1/\bar{y},\ 1/\underline{y}\right] \quad \text{if } 0 \notin y^I. \tag{2d}$$

Let f be a real-valued function of a single variable x, then the image of the set x^I under the mapping f is the interval-valued function given as

$$f\left(x^I\right) \triangleq \left\{f(x) \,|\, x \in x^I\right\} \tag{3}$$

which is an extension of the real-valued function $f(x)$ of the traditional real arithmetic. However, the results are different than those obtained via traditional arithmetic. For example, if $f(x) = x^2$, then it follows that [12]

$$\left(x^I\right)^2 = \begin{cases} \left[\underline{x}^2,\ \bar{x}^2\right] & \text{if } 0 \le \underline{x} \le \bar{x}; \\ \left[\bar{x}^2,\ \underline{x}^2\right] & \text{if } \underline{x} \le \bar{x} \le 0; \\ \left[0, \max\left(\underline{x}^2,\ \bar{x}^2\right)\right] & \text{if } \underline{x} < 0 < \bar{x}. \end{cases} \tag{4}$$

Furthermore, it is very disappointing to note that for the interval variable x^I the following results are obtained

$$x^I - x^I \ne 0; \tag{5a}$$

$$x^I / x^I \ne 1; \tag{5b}$$

$$x^I \times x^I \ne \left(x^I\right)^2 \quad \text{if } \underline{x} < 0 < \bar{x} \Rightarrow x^I \times x^I \supseteq \left(x^I\right)^2. \tag{5c}$$

These formulas show that unless interval computations are carried out with utmost care, results could turn out to be erroneous and misleading with respect to traditional real arithmetic. Moreover, the ordinary interval analysis often leads to an overestimation of the interval width that could be catastrophic from an engineering point of view. It can be easily proved that in interval analysis overestimation frequently occurs when an expression contains multiple instances of one or more interval variables. This is a consequence of traditional ordinary interval arithmetic operations which assume that the operand interval numbers are independent. Indeed, when the operands are partially dependent on each other, not all combinations of values in the given intervals will be valid and the exact interval will probably be smaller than the one produced by the formulas. This is the most evident result of the so-called *dependency phenomenon*. Sometimes, in the framework of interval arithmetic, it is possible to limit the overestimation of the interval width, which could lead to physically unacceptable results, by reducing the number of occurrences of each variable in the algebraic equations. However, more complex the function $f\left(x^I\right)$ is, the worse the dependency problem becomes. In particular, for long iterative computations, often a rapid growth of the interval at each stage is obtained.