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A new kind of regional importance measure of the input variable and its state dependent parameter solution

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ABSTRACT

To further analyze the effect of different regions within input variable on the variance and mean of the model output, two new regional importance measures (RIMs) are proposed, which are the "contribution to variance of conditional mean (CVCM)" and the "contribution to mean of conditional mean (CMCM)". The properties of the two RIMs are analyzed and their relationships with the existing contribution to sample variance (CSV) and contribution to sample mean (CSM) are derived. Based on their characteristics, the highly efficient state dependent parameter (SDP) method is introduced to estimate them. By virtue of the advantages of the SDP-based method, the same set of sample points utilized for solving CSM and CSV is enough to estimate CVCM and CMCM. Several examples demonstrate that CVCM can provide further information on the existing CSV, which can effectively instruct the engineer on how to achieve a targeted reduction of the main effect of each input variable. CMCM can act as effectively as the CSM, but the convergence and stability for estimating CMCM by numerical simulation is better than those for estimating CSM. Besides, the efficiency and accuracy of the SDP-based method are also testified by the examples.

1. Introduction

Sensitivity analysis (SA), especially global SA, is widely used in engineering optimization, probability safety assessment and so on. Global SA, also known as importance analysis, focuses on determining which of the input variables affects output most in the whole uncertainty range of the inputs. Indicators created for global SA purpose are defined as how uncertainty in the output can be apportioned to different sources of uncertainty in the model input [1]. At present, many importance analysis techniques and indices are presented, such as nonparametric techniques [2–4], variance-based importance measure indices [5–8], and moment-independent importance measures [9–11]. Among these methods, variance-based importance measure is known as a versatile and effective tool in uncertainty analysis.

All the importance analysis techniques aim at identifying the important variable out of a set of input variables, i.e. obtaining the inter-variable importance. Once the most important input has been detected, these importance analysis techniques cannot tell which part of this important input, e.g. the left or the right tails, center region, near center, etc., contributes most to the output uncertainty. This means they cannot identify intra-variable importance. The knowledge of the critical region of an input variable can be very useful in engineering, since it can inform the engineer about how to act operatively in order to reduce the range of uncertainty of the important input for a given targeted reduction of the output uncertainty. Therefore, many studies have begun to focus on intra-variable importance. A localized probabilistic sensitivity method is proposed in Ref. [12] by Millwater et al. to indicate the regional importance of the input variables. Although this method can identify the intra-variable importance to some extent, the definition of the sensitivity index, in terms of the partial derivative of the probabilistic response with respect to the cumulative distribution function (CDF) of each input variable, is inappropriate. The reason is that the variation of an input variable's values cannot cause disturbance in its CDF, and the CDF is defined only by the distribution parameters of this variable. In 1993, Sinclair [13] introduced the contribution to the sample mean plot (CSM), which was further developed by Bolado–Lavin et al. [14]. CSM plot cannot only identify the inter-variable importance, but act as a powerful tool to localize regions of the input space that contribute substantially to the mean of the model output. In light of this, it is extended to the regional importance analysis of the input variables on the variance of the model output by Tarantola et al. [15], and a "contribution to sample variance plot" (CSV) is presented to identify the local regions of the input space where the contribution to the variance of the model output

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is considerable. This extension effectively instructs the engineer on how to achieve a targeted reduction of the variance by operating on the extremes of the input variables' ranges.

The CSV proposed in Ref. [15] only provides the effect of a particular region of input variable on the total variance of the model output, which cannot reflect the contribution of the internal regions of the input variable on the components of the total variance, i.e. the variance contribution of an individual input variable to the total variance in Sobol's variance decomposition. The component of the total variance is usually used to measure the importance of the input variable. Especially is the first order component of the total variance, also referred to as the main effect of the input variable, used more extensively in the practice. Knowledge about the critical regions of the input variables for their main effects can add more information directly to the variance-based global importance analysis. Therefore, a regional importance measure (RIM) is presented in this paper to represent the effect of the internal regionals of each input variable on its corresponding first order variance contribution. Similar as the definition of CSV, the presented RIM is described by the contribution to variance of conditional mean (CVCM), and the variance of conditional mean is the main effect of an individual variable to the global variance. The CVCM can effectively inform the engineer about how to achieve a targeted reduction of the main effect of each input variable, by trimming the range of this variable. Furthermore, the idea of the CVCM is then extended to define a new RIM, which is the contribution to mean of conditional mean (CMCM). It is shown by derivation of CMCM that CMCM has the same physical meaning as CSM. But the convergence and stability for estimating CMCM are better than those for CSM by the same sample. In theory, a "double-loop sampling" procedure is needed for the calculation of the proposed RIMs, however the efficiency of double-loop sampling is very low. To overcome this difficulty, a state dependent parameter (SDP) method [16–19] is employed to solve CVCM and CMCM. The results show that the same set of sample points employed by the CSM and CSV is enough to solve CVCM and CMCM. Therefore, more referential information can be obtained without additional model evaluations.

The remainder of the paper is organized as follows: In Section 2 the definitions of the CSM and CSV are reviewed. Two new RIMs are defined and their properties are derived and proved in Section 3. The relationships between the proposed RIMs and the existing CSV and CSM are explored and discussed in Section 4. Section 5 focuses on introducing the high efficient SDP method to solve the regional importance analysis. The proposed RIMs are applied to several numerical and engineering examples in Section 6. The results of CVCM and CMCM are discussed and compared with those of CSM and CSV, which can demonstrate the effectiveness of the proposed RIMs. The accuracy and efficiency of the proposed SDP solutions of the presented RIMs are also testified in this section. Finally, the conclusion comes at the end of the paper.

2. Review on the CSM and CSV plots

Considering a mathematical or computational model with the form $y = g(\boldsymbol{x})$, where y is the model output, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ is the n-dimensional independent input variables with uncertainty. The definition of CSM for a given input x_i is as follows [14]:

$$CSM_{x_i}(q) = \frac{1}{E(y)} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \int_{-\infty}^{F_i^{-1}(q)} \prod_{j=1}^{n} f_j(x_j) g(x_1, x_2, \ldots, x_n) dx_i dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n$$
$$E(y) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{j=1}^{n} f_j(x_j) g(x_1, x_2, \ldots, x_n) dx_1 \ldots dx_n \tag{1}$$

where quantile $q \in [0, 1]$. $E(y)$ is the mean of the model output. $F_i^{-1}(q)$ is the inverse CDF of x_i at quantile q. $f_j(x_j)$ is the probability density function (PDF) of the jth input variable x_j. The multiple integral in Eq. (1) is calculated on the range $[-\infty, +\infty]$ for all the input variables except for x_i, for which the range is $[-\infty, F_i^{-1}(q)]$, i.e. the mean value used in Eq. (1) is conditioned on the range of x_i.

Since q is a point on x-axis representing a fraction of distribution range of x_i, $CSM_{x_i}(q)$ is a fraction of the output mean due to the values of x_i smaller or equal than its q-quantile. It is worth mentioning that fraction is only used for non-negative random variables here and in the sequel. The CSM plot for input variable x_i can be obtained by plotting the $CSM_{x_i}(q)$ against q, i.e. the CDF of x_i. Note that when the model response does not take only positive values, transformations are required in order to ensure the reliability of the approach, i.e. ensure that $CSM_{x_i}(q) \in [0, 1]$ [14].

Similarly, the CSV for a given input x_i is defined as [15]:

$$CSV_{x_i}(q) = \frac{1}{V(y)} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \int_{-\infty}^{F_i^{-1}(q)} \prod_{j=1}^{n} f_j(x_j) (g(x_1, x_2, \ldots, x_n) - E(y))^2 dx_i dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n$$
$$V(y) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{j=1}^{n} f_j(x_j) (g(x_1, x_2, \ldots, x_n) - E(y))^2 dx_1 \ldots dx_n \tag{2}$$

where $V(y)$ is the variance of the model output.

According to the definition, $0 \le CSV_{x_i}(q) \le 1$, and $CSV_{x_i}(q)$ represents a fraction of the output variance due to the values of x_i smaller or equal than its q-quantile. The CSV plot for x_i is obtained by plotting the $CSV_{x_i}(q)$ against q. It is important to note that the CSV is defined in Eq. (2) as a contribution to variance with respect to constant mean $E(y)$ over the full range of all parameters [15].

While the CSM allows identifying the effect of each input variable on the average of the model output for any given percentile within its uncertainty range, the CSV does the same for the output variance.

The properties of the CSM and CSV plots could be found in Refs. [14] and [15].

3. Two new RIMs and their properties

3.1. The definitions of the two new RIMs

Importance analysis based on variance is to quantify the contribution of each input variable to the output variance. It is related to the variance decomposition equation of Sobol [5]:

$$V(y) = \sum_i V_i + \sum_i \sum_{j>i} V_{ij} + \ldots + V_{1,2,\ldots,n} \tag{3}$$

where V_i is the first order variance contribution of the ith input variable x_i, and can be computed by

$$V_i = V(E(y|x_i)) \tag{4}$$

$$V_{ij} = V\big(E(y|x_i, x_j)\big) - V_i - V_j$$
$$V_{ijk} = V\big(E(y|x_i, x_j, x_k)\big) - V_{ij} - V_{ik} - V_{jk} - V_i - V_j - V_k \tag{5}$$

are higher order variance terms which reflect the contribution to the output variance by interaction of input variables in the model function. When only the first order variance contribution dominates, the variance decomposition equation can be rewritten as

$$V(y) = \sum_{i=1}^{n} V_i \tag{6}$$

The first order variance contribution V_i is also referred to as the main effect of x_i on the output variance [5], and can describe the importance of input variables to a certain extent. Actually, a lot of research has been focused on the computation and quality of V_i. In this work, we first also focus on the main effect, and propose a new RIM to study the effect of the internal regions of each input variable on its main effect, and the results can be easily extended to the regional importance analysis of the input variables on their other effects.