Reliability Engineering and System Safety 109 (2013) 119–122

Contents lists available at SciVerse ScienceDirect

Reliability Engineering and System Safety

journal homepage: www.elsevier.com/locate/ress

Short communication

On dependent items in series in different environments

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ARTICLE INFO

Article history:
Received 30 March 2012
Received in revised form
12 August 2012
Accepted 16 August 2012
Available online 27 August 2012

Keywords:
Accelerated life model
Proportional hazards model
Bivariate distribution
Series system
Competing risks

ABSTRACT

The Accelerated Life Model (ALM) and the proportional hazards (PH) model are very popular in reliability theory and applications as tools for modeling an impact of another environment on reliability characteristics of items defined for some baseline environment. These models were extensively studied for single items or systems. However, generalizations of the ALM and the PH model to the case of possibly dependent items were not considered in reliability literature as a tool for modeling the impact of environment. In this note, we suggest and discuss possible approaches for defining bivariate ALM and PH models and consider several examples.

1. Introduction

The Accelerated Life Model (ALM) and the proportional hazards (PH) model are very popular in reliability theory and applications as convenient tools for modeling, e.g., an impact of a more severe environment on reliability characteristics of items defined for some baseline environment. These models were extensively studied in the literature for single items or systems (see, e.g., Refs. [1–5] to name a few). Specifically, numerous practical results dealing with statistical inference and accelerated life testing were reported. However, generalizations of the ALM and the PH model to the case of possibly dependent items, which can be meaningful for reliability practice, are not trivial and therefore, challenging. We feel that the as suggested in this note new bivariate models have a potential for further development and applications. In the current section, in view of further generalizations, we first deal with the univariate case and discuss some relevant basics [6].

Consider one degrading item which operates in a baseline environment (regime) and denote the corresponding Cdf of time to failure T_b by $F_b(t)$. By "degrading" we mean that the failure occurs due to some degradation processes, e.g., as a result of wear accumulation. Let another statistically identical item described by the lifetime T_s with the Cdf $F_s(t)$ be operating in a more severe environment. Assume for simplicity that environments are not varying with time and that distributions are absolutely continuous and denote by $\lambda_b(t)$ and $\lambda_s(t)$ the corresponding failure rates. It is reasonable to assume that degradation in the second regime is more intensive and therefore, the time for accumulating the same amount of degradation or wear is smaller than in the baseline one. Therefore, assume that the corresponding lifetimes are ordered in terms of the (usual) stochastic ordering [7] as

$$S_s(t) \equiv \overline{F}_s(t) \leq \overline{F}_b(t) \equiv S_b(t),\ t \in (0,\infty). \quad (1)$$

This general relationship naturally models the impact of a more severe environment as compared with the baseline one. Note that, the corresponding failure rate ordering

$$\lambda_b(t) \leq \lambda_s(t),\ t \in (0,\infty), \quad (2)$$

which is also very popular in reliability applications, obviously, is a more stronger one and, therefore, leads to (1).

Inequality (1) implies the following equation:

$$F_s(t) = F_b(W(t)),\ W(0) = 0,\ t \in (0,\infty), \quad (3)$$

where the *scale transformation function* $W(t) > t$ is strictly increasing. This follows after applying the inverse function to both parts of (3), i.e., $W(t) = F_b^{-1}(F_s(t))$ and noting that the superposition of two increasing functions is also increasing. Thus, Eq. (3) *can be interpreted* as the general ALM [6,8] with the scale transformation function $W(t)$.

Remark 1. It should be noted that the ALM (3) in the described context is a formal consequence of ordering (1), whereas in reliability applications, only the specific linear case $W(t) = wt$ is usually considered. However, this linear relationship and the ALM defined via it can be justified at some instances by considering the physics of failure and deterioration [9,10].

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http://dx.doi.org/10.1016/j.ress.2012.08.005

The function $W(t)$ is differentiable, therefore it can be interpreted as an additive (relative) degradation function: $W(t)=\int_0^t w(u)du$, where $w(t)$ in its turn can be interpreted as the degradation rate [11,12]. Without losing generality, we assume for convenience that in the baseline environment, the degradation rate is equal to 1.

On the other hand, inequality (2) implies

$$\lambda_s(t)=k(t)\lambda_b(t),\ t\in(0,\infty), \tag{4}$$

which is a time-dependent proportional hazards (PH) model.

Thus (3) and (4) *formally* define the two most popular patterns for modeling the impact of environment on a single item. In reliability practice, however, only linear $W(t)=wt$ or $\lambda_s(t)=k\lambda_b(t)$, $k>0$, are usually employed, as the corresponding statistical analysis can be effectively performed in this cases.

It follows from the exponential formula of reliability, $F(t)=1-\exp\{-\int_0^t \lambda(u)du\}$ that when the failure rates are given, or estimated from the data, the ALM (3) can be viewed as an equation for obtaining $W(t)$ (uniquely):

$$\int_0^t \lambda_s(u)du=\int_0^{W(t)} \lambda_b(u)du. \tag{5}$$

Example 1. Let both failure rates be linear: $\lambda_b(t)=k_1t$, $\lambda_s(t)=k_2t$; $k_2>k_1$. It follows from (5) that $W(t)=\sqrt{k_2/k_1}\ t$ and we see that the linear PH model corresponds to the linear ALM in this specific case.

Our goal in this note is to generalize these models to the series system of $2(n)$ possibly dependent items. In the next section, we will briefly discuss the straightforward case of statistically independent items and will continue with a non-trivial case of statistically dependent items in Section 3. Finally, some comments will be given in Section 4.

2. Independent items in series

Survival functions of a series system of n statistically independent items (competing risks) under the baseline and a more severe environment, in accordance with (3), are

$$\overline{F}_b(t)=\prod_1^n \overline{F}_{bi}(t),\ \ \overline{F}_s(t)=\prod_1^n \overline{F}_{bi}(W_i(t)), \tag{6}$$

respectively, where $W_i(t)$ is the scale transformation function for the ith item. Thus $W(t)$for the system can be obtained from the following equation:

$$F_b(W(t))=\prod_1^n F_{bi}(W_i(t)) \tag{7}$$

or, equivalently, using relationships similar to (5):

$$\int_0^{W(t)} \sum_1^n \lambda_{bi}(u)du=\sum_1^n \int_0^{W_i(t)} \lambda_{bi}(u)du. \tag{8}$$

Example 2. Finkelstein [6]. Let $n=2$. Let $W_1(t)=t$, $W_2(t)=2t$, which means that the first component is protected from the more severe environment. Then Eq. (8) turns to

$$\int_0^{W(t)} (\lambda_{b1}(u)+\lambda_{b2}(u))du=\int_0^t \lambda_{bi}(u)du+\int_0^{2t} \lambda_{b2}(u)u.$$

Assume further that the failure rates a linear $\lambda_{b1}(t)=\lambda_1 t$, $\lambda_{b2}(t)=\lambda_2 t$, $\lambda_1,\lambda_2>0$. Then

$$W(t)=\left(\sqrt{\frac{\lambda_1+4\lambda_2}{\lambda_1+\lambda_2}}\right)t$$

If the components are statistically identical in the baseline environment ($\lambda_1=\lambda_2$), then $W(t)=\sqrt{5/2}\ \ t\approx 1.6t$.

For the PH model, due to independence, it obviously follows from (4) that for each item

$$\lambda_{si}(t)=k_i(t)\lambda_{bi}(t),\ t\in(0,\infty),$$

whereas for the series system, assuming the time-independent impact of a more severe environment on the failure rates of items in the baseline environment, we have

$$\lambda_s(t)=\sum_1^n k_i\lambda_{bi}(t) \tag{9}$$

3. Dependent items in series

3.1. Generalization of the univariate ALM

What happens when our items are statistically dependent? We will consider for simplicity of notation the case of two components, $n=2$. Before generalizing ALM to this case, we first describe the dependence of components via the concept of *copula.* A formal definition and numerous properties of copulas can be found, *e.g.*, in [13]. Copulas create a convenient way of representing multivariate distributions. In a way, they 'separate' marginal distributions from the dependence structure. It is more convenient for us to consider the survival copulas based on marginal survival functions [13]. In order to deal with the series system (competing risks), we must first consider the general bivariate ($n=2$) case. For $n>2$, the discussion is similar.

Let $T_{b1}\geq 0, T_{b2}\geq 0$ be the possibly dependent lifetimes of items in the baseline environment and let

$$F_b(t_1,t_2)=\Pr[T_{b1}\leq t_1,\ T_{b2}\leq t_2],$$
$$F_{bi}(t_i)=\Pr[T_{bi}\leq t_i],\ i=1,2,$$

be the absolutely continuous bivariate and univariate (marginal) Cdfs, respectively, in the baseline environment. The similar notation with the sub index "s" is for the more severe environment. Denote the bivariate (joint) survival function by

$$S_b(t_1,t_2)\equiv \Pr[T_{b1}>t_1,\ T_{b2}>t_2]=1-F_{b1}(t_1)-F_{b2}(t_2)+F_b(t_1,t_2) \tag{10}$$

and the univariate (marginal) survival functions with the corresponding failure rates $\lambda_{bi}(t_i), i=1,2$ by

$$S_{b1}(t_1)\equiv \Pr\ [T_{b1}>t_1,\ T_{b2}>0]=\Pr[T_{b1}>t_1]=S_b(t_1,\ 0),$$
$$S_{b2}(t_2)\equiv \Pr\ [T_{b1}>0,\ T_{b2}>t_2]=\Pr[T_{b2}>t_2]=S_b(0,\ t_2),$$

It is well-known [13] that the bivariate survival function $S_b(t_1,t_2)$ can be represented as a function of $S_{bi}(t_i), i=1,2$ in the following way:

$$S_b(t_1,\ t_2)=C(S_{b1}(t_1),\ S_{b2}(t_2)), \tag{11}$$

where the survival copula $C(u,v)$ is a bivariate function in $[0,1]\times[0,1]$. Note that, such a function always exists when the inverse functions for $S_{bi}(t_i),\ i=1,2$ exist:

$$S_b(t_1,\ t_2)=S_b(S_{b1}^{-1}S_{b1}(t_1),\ S_{b1}^{-1}S_{b2}(t_2))=C(S_{b1}(t_1),\ S_{b2}(t_2)).$$