Ocean Engineering 119 (2016) 58–66

Contents lists available at ScienceDirect

Ocean Engineering

journal homepage: www.elsevier.com/locate/oceaneng

Three dimensional sloshing of stratified liquid in a cylindrical tank

Zhen Wang [a,*], Li Zou [b], Zhi Zong [b]

[a] School of Mathematical Science, Dalian University of Technology, Dalian 116024, China
[b] School of Naval Architecture, Dalian University of Technology, Dalian 116085, China

ARTICLE INFO

Article history:
Received 23 January 2016
Received in revised form
15 March 2016
Accepted 24 April 2016
Available online 3 May 2016

Keywords:
Continuous stratified liquid
Resonance
Sloshing
Cylindrical tank

ABSTRACT

The sloshing of stratified liquid with density varying with depth in a three dimensional cylindrical tank is considered in the framework of linearized theory. The flow of stratified liquid is no longer irrotional and the governing equation is no longer Laplacian. The stream function is also invalid for three dimensional sloshing, which is different from two dimensional sloshing. We adopt the governing equations and boundary conditions in terms of a pressure function and a density function to instead of velocity potential for uniform liquid. Separation of variables and Laplace transformation methods are used to solve the governing equations for the constant Brunt–Väisälä frequency. The Residue theorem is applied to calculate the inverse Laplace transformation and the resonant behavior is also analyzed. It is found that the natural frequencies are weakened by the stratification of the liquid density. New natural frequencies have appeared which do not exist for uniform liquid. We find that these new frequencies have some special characters; they make the motion history of free surface irregular, the amplitudes caused by new frequencies are remarkable, nearly two-thirds of the amplitude caused by natural frequencies and all new frequencies are less than Brunt–Väisälä frequency *N*.

1. Introduction

Sloshing is a classic and important problem of fluid dynamics. It also has a wide range of applications in engineering, such as moving vehicle and structure containing a liquid with a free surface, ships, offshore platform, harbor resonance. Resonant motion of the moving liquid can become extremely large if no or weak hydrodynamics damping is present. Sloshing is important for the safety of the structures (Faltinsen and Timokha, 2009; Lighthill, 2001; Yih, 1965; Batchelor, 1967). There are many publications devoted in sloshing phenomenon, but most of them focus on the case of uniform density and two layer liquid (Faltinsen et al., 2011; Tang, 1993b; Veletsos and Shivakumar, 1995; Gavrilyuk et al., 2005; Ardakanin et al., 2015; Sinai, 1985; Hiroyuki and Seiichi, 1985).

Sloshing of liquid with variable density is also an important issue and applicable in engineering, such as waste storage tanks and crude oil storage tanks. Bandyopadhyay (1991) indicated that a large number of high level waste (HLW) storage tanks at various U.S. Department of Energy (DOE) facilities contain liquid with nonuniform density. After prolonged storage, the waste material and the crude oil gradually deposit on the bottom of the tank. It may even form continuous variation of sludge at the tank base. So the difference of the density can be large from top to bottom of the tank. When gas is mixed into the liquid, the liquid can also be regarded as stratified liquid. In fact, there are also waves in the inner of the stratified liquid, which are called internal waves (Valentine, 2005).

Sloshing of continuously stratified liquid has its own characters and phenomenons. The predictions obtained through uniform density assumption may no longer hold. Thus, there is a need to understand the effect of nonuniform density on the dynamics response of the contained liquid. There are relatively less results for continuous stratified fluids. Most works assumed to be composed of several layers with piecewise uniform densities (Faltinsen and Timokha, 2009; Tang, 1993a,b). This simplification introduces a pressure jump condition or shear stress at the interface of adjacent layers (Faltinsen and Timokha, 2009). Exploratory studies on the dynamic response of tanks containing two liquids have been performed by Tang (1993a,b). Those studies show that the dynamic response of a tank containing two liquids is quite different from that of an identical tank containing only one liquid. The sloshing wave height may increase significantly in a tank that contains two liquids. Therefore, it is necessary to understand the sloshing response of tanks that contain liquid with nonuniform density in order to design and evaluate the HLW storage tanks.

Many different numerical methods are developed for sloshing, such as Chapman and Porter (2005), Robertson et al. (2004), Frandsen (2004), Kisheve et al. (2006), and Wu et al. (1998). The nonlinear resonance are also studied by numerical methods in Wu

* Corresponding author.
E-mail addresses: wangzhen@dlut.edu.cn (Z. Wang), lizou@dlut.edu.cn (L. Zou).

http://dx.doi.org/10.1016/j.oceaneng.2016.04.028
0029-8018/

(2007) and Zhang (2014). The natural sloshing modes are very important for studying the sloshing phenomenon. There are also some works on sloshing modes and frequencies, such as Faltinsen and Timokha (2012a,b, 2014), McIver (1989), and Scolan (2015). An analytical solution for a liquid-filled tank with continuously varying density is a difficult task. Liquid with variational density continuously is no longer governed by irrotional and Laplacian type governing equation. For two dimensional rectangular tank, Wu (2011) adopted stream function to analyze the sloshing, but stream function method does not work for the three dimensional sloshing. In this paper, we focus on the motion of the incompressible liquid with vertical stratified density in a three dimensional cylindrical tank. We will deduce a differential equation in terms of pressure to represent the motion of stratified liquid with the help of the linearized momentum equations and the continuity equation. The governing equation in terms of pressure is different from that expressed by velocity potential when the density is constant. The boundary conditions are also expressed for the pressure, and then the Laplace transformation and separation of variables are applied to solve the problem.

The behavior of the motion of the stratified liquid is also different from that of uniform liquid. Motion of the stratified liquid leads to a more complicated equation for natural frequencies, and the dispersion relation is also more complicated. However it can be reduced to the result of constant density liquid.

2. Governing equations and boundary conditions

The motion of stratified liquid in a circular cylindrical tank with depth d and radius R_0 is considered. Cylindrical coordinate system (r, θ, z) is adopted, the origin is located at the center of the static free surface and z-axis points upward which is the same in both coordinate systems.

The liquid is assumed to be inviscid. We also assume that the density depends on the depth of the fluid, in other words, the density $\rho_0(z)$ is a function of z. When the fluid is set into motion, we divide the density into static part $\rho_0(z)$ and dynamic part $\rho'(x, y, z, t)$, i.e. $\rho(x, y, z, t) = \rho'(x, y, z, t) + \rho_0(z)$. In what follows, we will denote $\rho_0(z)$, $\rho(x, y, z, t)$ and $\rho'(x, y, z, t)$ by ρ_0, ρ and ρ', respectively. For still water, the velocities are set $\mathbf{v} = (u, v, w) = 0$. The magnitude of static density ρ_0 is assumed as $O(1)$. The magnitudes of dynamics density ρ', velocities $\mathbf{v} = (u, v, w)$ and pressure function $p = p(x, y, z, t)$ for small motion are assumed as $O(\epsilon)$. Momentum equation can be expressed by

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\ \mathrm{grad}\ p + \mathbf{F},$$

where $\mathbf{F} = (0, 0, -g)^T$, g is the gravitational acceleration. Linearized momentum equation for small motion can be written as

$$u_t = -\frac{1}{\rho}p_x, \quad v_t = -\frac{1}{\rho}p_y, \quad w_t = -\frac{1}{\rho}p_z - g. \tag{1}$$

Since $\rho = \rho_0 + \rho'$, then the linearized momentum equation can be rewritten as

$$\rho_0 u_t = -p_x, \quad \rho_0 v_t = -p_y, \quad \rho_0 w_t = -p_z - (\rho' + \rho_0)g. \tag{2}$$

Here if ρ is constant, then the liquid is irrotional and velocity potential function could be used to analyze sloshing of constant density liquid. But now ρ_0 is a function of z, then the rotation of velocity $\nabla \times \mathbf{v}$ is not always equal to zero. A different procedure is necessary.

Since sloshing of the liquid is considered in this paper, we can assume the liquid is incompressible as mentioned above, it means that

$$\frac{d\rho}{dt} = \rho_t + u\rho_x + v\rho_y + w\rho_z = 0, \tag{3}$$

which can be found in many contexts, such as Batchelor (1967). In the sense of linear approximation, it has a simple form by neglecting nonlinear terms

$$\rho_t' + \rho_{0z} w = 0. \tag{4}$$

Velocity component w can be replaced by pressure p with the aid of linearized momentum equation in z component in Eq. (2), then the incompressible condition can be expressed by pressure and density as follows:

$$\rho_{tt}' - \frac{\rho_{0z}}{\rho_0}p_z - g\frac{\rho_{0z}}{\rho_0}(\rho' + \rho_0) = 0. \tag{5}$$

We note that Eq. (5) has the same form under the cylindrical coordinate.

Since the liquid is incompressible as shown in Eq. (3), the continuity equation

$$\frac{d\rho}{dt} + \rho(u_x + v_y + w_z) = 0$$

takes the form as

$$u_x + v_y + w_z = 0. \tag{6}$$

Multiplying Eq. (6) with ρ_0 and differentiating Eq. (6) with respect to t, we get an equation in terms of pressure p and dynamic density ρ' by the aid of Eq. (2)

$$\nabla^2 p + \rho_z' g - \frac{\rho_{0z}}{\rho_0}(p_z + \rho' g) = 0, \tag{7}$$

which also can be rewritten as

$$p_{zz} + \frac{1}{r}(rp_r)_r + \frac{1}{r^2}p_{\theta\theta} + \rho_z' g - \frac{\rho_{0z}}{\rho_0}(p_z + \rho' g) = 0. \tag{8}$$

in cylindrical coordinates (r, θ, z).

Eqs. (8) and (5) are the governing equations in the whole zone of the liquid under cylindrical coordinate. These equations are in terms of pressure function and density function that deduced from the momentum equation (Eq. (2)), continuity equation (Eq. (6)) and incompressible condition (Eq. (3)).

The dynamics boundary condition on the free surface $z = \eta(x, y, t)$ can be written as

$$p(x, y, z = \eta, t) = 0. \tag{9}$$

We will deduce a condition for pressure at mean level z=0. In order to match the governing equations which are in terms of pressure and density for varying density flow, linear approximation of the pressure at the mean surface level z=0 gives

$$p(x, y, 0, t) + \frac{\partial p(x, y, 0, t)}{\partial z}\eta = 0, \tag{10}$$

where $\frac{\partial p(x, y, 0, t)}{\partial z}$ can be given by $-\rho_0(0)g$ with the aid of the third momentum equation of Eq. (2), which also called hydrostatic condition (Yih, 1965). So Eq. (10) can be written as

$$p(x, y, 0, t) - \rho_0(0)g\eta = 0. \tag{11}$$

This relation also gives a direct link between pressure and free surface. The expression of free surface will be given by the solution of pressure in next section.

Differentiating Eq. (11) with respect to t twice, and considering the relation $\eta_t = w$ and linear momentum equation on the free surface z=0, we get

$$p_{tt} + gp_z + g^2(\rho' + \rho_0(0)) = 0 \text{ on } z = 0. \tag{12}$$