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# Reliability of mechanisms with periodic random modal frequencies using an extreme value-based approach

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ABSTRACT

Resonance in a dynamic system is to be avoided since it often leads to impaired performance, overstressing, fatigue fracture and adverse human reactions. Thus, it is necessary to know the modal frequencies and ensure they do not coincide with any applied periodic loadings. For a rotating planar mechanism, the coefficients in the mass and stiffness matrices are periodically varying, and if the underlying geometry and material properties are treated as random variables then the modal frequencies are both position-dependent and probabilistic. The avoidance of resonance is now a complex problem. Herein, free vibration analysis helps determine ranges of modal frequencies that in turn, identify the running speeds of the mechanism to be avoided. This paper presents an efficient and accurate sample-based approach to determine probabilistic minimum and maximum extremes of the fundamental frequencies and the angular positions of their occurrence. Then, given critical lower and upper frequency constraints it is straightforward to determine reliability in terms of probability of exceedance. The novelty of the proposed approach is that the original expensive and implicit mechanistic model is replaced by an explicit meta-model that captures the tolerances of the design variables over the entire range of angular positions: position-dependent eigenvalues can be found easily and quickly. Extreme-value statistics of the modal frequencies and extreme-value statistics of the angular positions are readily computed through MCS. Limit-state surfaces that connect the frequencies to the design variables may be easily constructed. Error analysis identifies three errors and the paper presents ways to control them so the methodology can be sufficiently accurate. A numerical example of a flexible four-bar linkage shows the proposed methodology has engineering applications. The impact of the proposed methodology is two-fold: it presents a safe-side analysis based on free vibration methods to assess and manage the uncertainty in the range of modal frequencies, and, it provides a launching platform for timely design optimization.



## 1. Introduction

The study of modal frequencies of structures and mechanisms helps ensure that the frequency of any applied periodic loading will not coincide with a modal frequency which may lead to impaired performance, overstressing, fatigue fracture and adverse human reactions due to resonance. The equations of motion ultimately provide the response of interest; however, in free vibration analysis, the motion-dependent terms associated with damping and gyroscopic terms are dropped from the global equations of motions, leaving only the mass and stiffness terms in the differential equations. These equations lead directly to an eigen-analysis to determine modal frequencies. Indeed all studies in dynamics and stability invoke an eigen-analysis at some level. For example, an eigen-analysis can be invoked in dynamics to translate nodal displacements into modal variables [1]. Herein we are interested in stability.

In planar linkage mechanisms such as the four-bar linkage, slider crank and the six-bar crank-rocker, the use of finite element methods is common to provide a finite number of degrees of freedom and thus employs lumped mass and lumped stiffness matrices [2–8]. Since the mass and stiffness of such systems are periodic functions due to the rotational effects, then to determine

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the dynamic modal frequencies and mode shapes, it is common to consider large scale rigid-body motion and small elastic longitudinal and lateral deflections modelled as two-dimensional beams [9,10]. The position-dependent natural frequencies can be determined from the following standard eigenvalue equations for instantaneous rigid-body configurations

$$\left(\mathbf{K}(\mathbf{v},\theta)-\omega_k^2(\mathbf{v},\theta)\mathbf{M}(\mathbf{v},\theta)\right)\boldsymbol{\psi}_k(\mathbf{v},\theta)=0 \tag{1}$$

where $\omega_k$ is the $k_{\text{th}}$ order natural frequency of the structure, $\boldsymbol{\psi}_k$ is the corresponding eigenvector and $\mathbf{v}$ is a vector of design variables that include material properties and physical parameters. Typically, the continuous position parameter $\theta\in[0,2\pi]$ is allotted discrete increments and then each position $\theta_j$ is evaluated as

$$\theta_j=\frac{2\pi(j-1)}{c-1}\text{ with } j=1,2\ldots c \tag{2}$$

The angular positions are stored in a vector $\boldsymbol{\theta}$. The harmonic compositions of the natural frequencies are often obtained by fast Fourier Transforms, wherein the number of sampling points $c$ provides the required accuracy [1]. This rule holds in the present work as well.

A mechanism may become unstable when the input rotates at speeds falling within unstable ranges and it follows that deformations, strains and stresses may become significant in any of the links. There are two important problems to address: The first is the ability to analyze the mechanism to identify the undesirable ranges that may originate from (a) the forced excitations from the drivers and (b) the so-called parametric excitations that originate due to variation in mass, stiffness and so forth. The second problem is the ability – given the operating ranges-to design the mechanism to operate satisfactorily in these designated ranges.

In this paper we focus on the second problem above and so critical specified frequencies are important. For example, Xiamin and Yunwen [11] consider mechanisms with multiple frequency constraints and Kalaycioglu and Bagci [12] introduced a simple method for determining the critical running speed associated with forced resonance based simply on the minima of natural frequencies during a cycle of periodic motion. Their premise stated that if the mechanism is driven at a constant input speed, then the maximum torque (and maximum power) occurs at the position that gives the minimum value of the fundamental frequency. They deduced that the minimum value of the fundamental frequency is critical because if the mechanism moves at a corresponding speed the inertial forces and torques (although periodic) will have the same frequency as the frequency of the motion and excite the mechanism at each cycle and gradually force the mechanism into resonance if damping is found wanting.

Although a deterministic analysis provides useful information, the uncertainty in the components (e.g. tolerances, material properties and degradation) must be considered for a more complete picture; That is, the uncertainties introduce system errors that affect the performance of the mechanism that manifest as (i) uncertain position-range responses and (ii) uncertain modal frequencies.

The uncertain position-range of the mechanism has been studied recently and gives direction to the work herein. The uncertain position-range can be evaluated as either (i) a point-reliability, or (ii) a system reliability. For a point-type reliability, probability is evaluated for some response at a particular time or position, whereas for the system-type reliability, probability encases a wide range of times or positions [13]. To evaluate the point reliability, Kim et al [14] applied the first-order reliability method (FORM) to compute the reliability of a rigid mechanism with normally distributed variables. Wu and Rao [15] discussed optimal allocation of tolerances using interval analysis. Huang and Zhang [16] applied Taguchi's robust design method to determine optimal specifications under the constraints of positional accuracy and assembly cost.

System reliability of mechanisms is computationally demanding, because the system failure region is represented by multiple correlated performance functions [17]. Literature in the system reliability analysis of rotating mechanism is rather limited; however, Zhang and Du [18] utilized a stochastic process to describe the outputs along the whole cyclic range of performance. This study was based on the assumption that the counters of the occurrence of a failure event in disjoint time intervals follow the independent Poisson distribution [19,20]. Given the point reliability indices, the calculation of the crossing-rate with respect to the threshold limit was used to envision a parallel system [21,22]. Another option is the extreme value method [23]. The one-dimensional extreme value distribution retains all of the correlation information of the original random variables. A failure is said to occur over an interval if the performance measure is greater than, or less than, a specified threshold at any time or position over the interval. The extreme value method uses the global maximum or minimum of the performance measure. Thus a failure event is equivalent to the event that the extreme value over the interval is greater than, or less than, the specified threshold. Therefore if the distribution of the extreme value can be identified, then the simpler point reliability method can be used to solve the time-dependent reliability problem. This method has been applied to analyze the time-dependent kinematic reliability of a rigid four-bar function generator. More specifically, Pandey and Zhang [24] applied a small number of samples to estimate fractional moments of the system error along a whole output trajectory. A probability distribution of the maximal output error, then, was determined by using the principle of maximum entropy. Additionally, the method was employed for system kinematics reliability analysis of rigid robotic manipulators [25].

The study of periodic randomness of natural frequencies in planar mechanisms is not so prolific, although several methods are available to find the random eigenvalue-vector pair for non-periodic systems. See, for example Rahman [26] and Fricker et al [27]. The uncertain and periodic mass and stiffness matrices give a position-dependent random eigenvalue problem. The study of the probabilistic natural frequencies over the entire time or position-range, instead of a single instant, is important to avoid resonance. Although, Fourier series methods have been applied to help determine the natural frequencies of deterministic planar mechanisms, this approach would be computationally expensive when applied to uncertain planar mechanisms: this is due to the need for a large number of elastic degrees of freedom, a large number of experiments and a sufficiently large number of sampling points at discrete configurations to provide an acceptable accuracy [28].

The objective of the proposed study is to develop a computationally efficient and accurate method for the modal frequency reliability analysis of uncertain planar mechanisms based on free vibration. Since there will be multiple, implicit and correlated performance measures a methodology is needed to combine the system failure events. This large number of performance measures implies that the method of FORM is not adequate [29]. Further, the first-order second moment (FOSM) method is of limited applicability, as it is based on a spurious assumption of the normality of the output performance. It follows that a Monte-Carlo simulation (MCS) must be invoked. To enable MCS, a simple, explicit, response surface meta-model is constructed to replace the implicit mechanistic model. Now, eigenvalues of the rotating mechanism at arbitrary sets of variable at selected angles are easily and quickly calculated: the computational efficiency in the system reliability analysis is greatly increased. Error analysis suggests ways to control and minimize errors to ensure the method is sufficiently