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Analytical results for the skewness of the distribution of detector counts in a subcritical reactor

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A R T I C L E I N F O

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We show analytical results for the skewness of the distribution of counts of a detector inside a nuclear reactor. Like in the α-Feynman experiment, the departure of the skewness with respect to its Poisson value depends only on the correlation between counts and the detector efficiency and not on the source intensity, in addition its sensitivity to the correlation is larger. Therefore,without any extra experimental effort, the usual α-Feynman experiment can be improved with the analysis of the third moments.

1. Introduction

In a recent paper (Difilippo, 2015), a numerical analysis of the statistical nature of the detection process of a neutron detector in a subcritical reactor driven by a steady neutron source was made. The paper suggested the inclusion of third and fourth moments (skewness and kurtosis) to the usual analysis of the measurements of the distribution of counts in the detector as function of the gate time t. For this purpose the departures of the skewness and kurtosis with respect to Poisson statistics were defined. Further numerical analysis showed that the departure of the skewness with respect to the Poisson statistics were independent on the intensity of the source. This paper shows that this is analytically true.

2. Moments of the distribution of counts

This section summarizes the already published equations for the moments of the distribution of counts; in order to be non repetitive we rewrite these equation to the minimum in order to be self-contained, further details can be found in Difilippo (2015). The usual α-Feynman experiment measures the correlation introduced by the fission process comparing the variance (σ^2) to the mean ($\bar{r}$). The departure of the ratio from one $\sigma^2/\bar{r} \equiv 1+\psi$ is related to the detector efficiency and the dynamic parameters of the subcritical system and it is independent of the source intensity.

2.1. Moments of the distribution of counts via the probability distribution function

The elementary Markovian processes in our system and in time interval Δt, are: (1) the production of a neutron by the source of intensity S with probability $S\Delta t$ and (2) the fission, capture and detection events with probabilities $\Lambda_f\Delta t$, $\Lambda_c\Delta t$, and $\Lambda_d\Delta t$, per neutron. Denoting by $P(n,r,t)$ the probability of having at time t, n neutrons in the system and r counts in our detector, the probability distribution function is defined as

$$F(x,y,t) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} x^n y^r P(n,r,t) \quad (1)$$

From the probability balance equation it can be demonstrated (Pacilio, 1976) that $F(x,r,t)$ satisfies the equation,

$$\frac{\partial F}{\partial t} = S(1-x) + \left[\Lambda_c(1-x) + \Lambda_f(p(x)-x) + \Lambda_d(y-x)\right]\frac{\partial F}{\partial x} \quad (2)$$

in this equation

$$p(x) = \sum_{\nu_p=0}^{\infty} x^{\nu_p} p_{\nu_p} \quad (3)$$

where p_{ν_p} is the probability of the emission of ν_p prompt neutrons in the fission process. Equation (2) corresponds to the prompt neutron approximation which in general is a good one provided that time interval t is small in comparison with the decay times of the delayed precursors. Any moment of the distribution $P(n,r,t)$ can be calculated with the factorial moments which are solutions of a

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system of ordinary differential equations. More explicitly, the $(m+i)$ partial derivative of F with respect to x (m times) and with respect to y (i times) evaluated with Eq. (2) at $x = y = 1$ is

$$\left(\frac{\partial^{(m+i)}}{\partial x^m \partial y^i}\frac{\partial F}{\partial t}\right)_{x=y=1} = \langle n(n-1)(n-2)\dots(n-m+1)r(r-1)(r-2)\dots(r-i+1)\rangle \tag{4}$$

where the bracket indicate the average with the distribution $P(n,r,t)$.

2.2. System of differential equations to compute the variance and the skewness

The partial derivatives of $\partial F/\partial t$ in Eq. (4) are calculated according to the right hand side of Eq. (2). For the first moments we have

$$\frac{d\langle n\rangle}{dt} = S + \alpha\langle n\rangle \tag{5.1}$$

and

$$\frac{d\langle r\rangle}{dt} = \Lambda_d\langle n\rangle \tag{5.2}$$

where the prompt decay constant $\alpha = -\Lambda_a + \bar{\nu}_p\Lambda_f$, with $\Lambda_a = \Lambda_c + \Lambda_f + \Lambda_d$ and $\vec{\nu}_p$ the average number of prompt neutrons per fission.

Equations for the second factorial moments are

$$\frac{d\langle n(n-1)\rangle}{dt} = 2\alpha\langle n(n-1)\rangle + (2S + \Lambda_f\langle \nu_p(\nu_p-1)\rangle)\langle n\rangle \tag{6.1}$$

$$\frac{d\langle nr\rangle}{dt} = \alpha\langle nr\rangle + S\langle r\rangle + \Lambda_d\langle n(n-1)\rangle \tag{6.2}$$

and

$$\frac{d\langle r(r-1)\rangle}{dt} = 2\Lambda_d\langle nr\rangle \tag{6.3}$$

where $\langle \nu_p(\nu_p-1)\rangle = \sum_{\nu=0}^{\infty}\nu_p(\nu_p-1)p_{\nu_p}$.

Equations for the third factorial moments are

$$\frac{d\langle n(n-1)(n-2)\rangle}{dt} = 3\alpha\langle n(n-1)(n-2)\rangle + 3(S + \Lambda_f\langle \nu_p(\nu_p-1)\rangle \langle n(n-1)\rangle + \langle \nu_p(\nu_p-1)(\nu_p-2)\rangle\langle n\rangle \tag{7.1}$$

$$\frac{d\langle n(n-1)r\rangle}{dt} = 2\alpha\langle n(n-1)r\rangle + (2S + \Lambda_f\langle \nu_p(\nu_p-1)\rangle)\langle nr\rangle + \Lambda_d\langle n(n-1)(n-2)\rangle \tag{7.2}$$

$$\frac{d\langle nr(r-1)\rangle}{dt} = \alpha\langle nr(r-1)\rangle + 2\Lambda_d\langle n(n-1)r\rangle + S\langle r(r-1)\rangle \tag{7.3}$$

and

$$\frac{d\langle r(r-1)(r-2)\rangle}{dt} = 3\Lambda_d\langle nr(r-1)\rangle \tag{7.4}$$

3. Solutions for the factorial moments equation

There are nine coupled first order ordinary differential equations to calculate the factorial moments of the distribution of counts up to the third order; they can be solved numerically and in our simple steady case analytically.

3.1. Analytical solution

The equations are linear with constant coefficients and they can be solved sequentially in the written order, i.e. step i is the input to step $i+1$. Each equation is of the form

$$\frac{dy}{dt} + a(t)y = f(t) \tag{8}$$

with general solution (Pontryagin, 1962)

$$y(t) = y(0)e^{-g(t)} + e^{-g(t)}\int_0^t f(\tau)e^{g(\tau)}d\tau \tag{9}$$

where $g(t) = \int_0^t a(\tau)d\tau$.

The case of the steady state of a subcritical reactor with an external source S is of interest because of standard noise techniques. Long after the introduction of the source (in terms of $1/\alpha$) the steady values of $\langle n\rangle$, $\langle n(n-1)\rangle$ and $\langle n(n-1)(n-2)\rangle$ can be calculated by Eqs. (5.1), (6.1) and (7.1) by equating to zero the time derivatives:

$$< n >= S/\alpha_o \tag{10.1}$$

$$\langle n(n-1)\rangle = (S + \Lambda_f\langle \nu_p(\nu_p-1\rangle/2)S/\alpha_o \tag{10.2}$$

$$\langle n(n-1)(n-2)\rangle = \langle n(n-1)\rangle(S + \Lambda_f\langle \nu_p(\nu_p-1\rangle)/\alpha_o + \Lambda_f\langle \nu_p(\nu_p-1)(\nu_p-2)\rangle S/3\alpha_o^2 \tag{10.3}$$

where $\alpha_o = -\alpha$.

Once these variables are calculated, the different moments that include the number of counts r can be calculated analytically with the use of Eq. (9), providing zero initial conditions for the moments that include r, i.e. our detector system is reset each time we perform a measurement. The integration is straight forward, but because of the many constants that accumulate from the step by step solutions, one is the input for the following, the algebra is cumbersome; we show their interrelationship in Appendix. The main text discusses the functional forms and emphasizes the final result: the departure of the skewness with respect to the Poisson statistics is independent of the intensity of the source.

In this way we found in the literature the solution up to the second moment:

$$\langle nr\rangle = Bt + C\left(1 - e^{-\alpha_o t}\right) \tag{11.1}$$

and

$$\langle r(r-1)\rangle = at + bt^2 + c\left(1 - e^{-\alpha_o t}\right) \tag{11.2}$$

where the constants B, C, a, b and c are shown in Appendix.

From these equations the second central moment (the variance $\sigma^2 \equiv \langle r^2\rangle - \langle r\rangle^2 = \langle r(r-1)\rangle + \langle r\rangle - \langle r\rangle^2$) is related to the mean value $\langle r\rangle$ in the following way

$$\frac{\sigma^2}{\langle r\rangle} = 1 + \psi(t) \tag{12}$$

with

$$\psi(t) = A\left(1 - \frac{1 - e^{-\alpha_o t}}{\alpha_o t}\right) \tag{13}$$

where $\alpha_o = -\alpha$ and $A = \frac{\Lambda_d\Lambda_f\langle \nu_p(\nu_p-1)\rangle}{\alpha_o^2}$. For further notation use we define the function $F_y(t) \equiv \left(1 - \frac{1-e^{-\alpha_o t}}{\alpha_o t}\right)$.

The same analytical approach can be done for the third moments: Eqs. (7.2)–(7.4) for, respectively, $\langle n(n-1)r\rangle$, $\langle nr(r-1)\rangle$ and $\langle r(r-1)(r-2)\rangle$ are solved in that order. The results:

$$\langle n(n-1)r\rangle = A_1 t + A_2\left(1 - e^{-2\alpha_o t}\right) + A_3 e^{-\alpha_o t}\left(1 - e^{-\alpha_o t}\right) \tag{14}$$