Reliability Engineering and System Safety 131 (2014) 216–227

Contents lists available at ScienceDirect

Reliability Engineering and System Safety

journal homepage: www.elsevier.com/locate/ress

Estimation of the inverse Weibull distribution based on progressively censored data: Comparative study

Rola M. Musleh, Amal Helu *

Department of Mathematics, The University of Jordan, Amman 11942, Jordan

ARTICLE INFO

Article history:
Received 17 February 2014
Received in revised form
24 April 2014
Accepted 3 July 2014
Available online 11 July 2014

Keywords:
Inverse Weibull distribution
Approximate maximum likelihood
Least squares methods
Lindley's approximation
Progressive type-II censoring
Pitman nearness probability

ABSTRACT

In this article we consider statistical inferences about the unknown parameters of the Inverse Weibull distribution based on progressively type-II censoring using classical and Bayesian procedures. For classical procedures we propose using the maximum likelihood; the least squares methods and the approximate maximum likelihood estimators. The Bayes estimators are obtained based on both the symmetric and asymmetric (*Linex*, General Entropy and Precautionary) loss functions. There are no explicit forms for the Bayes estimators, therefore, we propose Lindley's approximation method to compute the Bayes estimators. A comparison between these estimators is provided by using extensive simulation and three criteria, namely, Bias, mean squared error and Pitman nearness (*PN*) probability. It is concluded that the approximate Bayes estimators outperform the classical estimators most of the time. Real life data example is provided to illustrate our proposed estimators.

1. Introduction

The Weibull distribution is one of the most popular and widely used models in life testing and reliability theory. Nevertheless, it has been found that the Weibull distribution does not provide a satisfactory parametric fit for those lifetime distributions with non-monotone failure rate, such as the unimodal failure rate functions [2,3,35, 36]. The density and the hazard function of the *IW* distribution can be unimodal or decreasing, depending on the choice of the shape parameter. Hence, if the empirical studies indicate that the hazard function may be unimodal, then the *IW* distribution is more appropriate model than the Weibull distribution [18]. The importance of the *IW* distribution can be emphasized in the following: The *IW* distribution provides a good fit to several data such as the time to breakdown of an insulating fluid subjected to the action of a constant tension [26]. Moreover, the *IW* distribution is a suitable model to describe the failure of the degradation phenomena of mechanical components of diesel engines such as pistons, crankshafts, and main bearings [16]. Extensive work has been done on the *IW* distribution using classical and Bayesian approaches, see for example [25,16,11,12,29,21,19] and recently [31].

If T follows $(\sim)$ a two-parameter Weibull distribution (α, β) with probability density function (pdf)

$$f(t;\alpha,\beta)=\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}e^{-(\alpha/t)^{-\beta}},\quad t>0,$$

then the failure time $X=1/T$ has an (IW) distribution with pdf

$$f(x;\alpha,\beta)=\alpha\beta(\alpha x)^{-\beta-1}e^{-(\alpha x)^{-\beta}},\quad x>0 \tag{1}$$

where $\alpha>0$ and $\beta>0$ are the scale and shape parameters respectively. If $X\sim IW(\alpha,\beta)$, then the distribution function of X is given by

$$F(x;\alpha,\beta)=e^{-(\alpha x)^{-\beta}},\quad x>0. \tag{2}$$

In life testing experiments, it is a common practice to cease testing before the failure of all items. This is due to the lack of funds and/or time constrains. Samples that result from such situations are called censored samples. There are several censoring methods available to experimenter, for example; type-I censoring in which the test ceases at a pre-fixed time, or type-II censoring that allows the experiment to be terminated at a predetermined number of failures. These methods do not allow the removal of active units during the experiment, therefore, the focus in the last few years has been on progressive censoring due to its flexibility that allows the experimenter to remove active units during the experiment.

A progressively type-II censoring is a generalization of type-II censoring. Under this type, n independent items are placed at the same time on a life testing experiment and only m $(<n)$ failures are completely observed. The censoring occurs progressively in m stages as follows: When the first failure is observed, a random sample of size R_1 is immediately drawn and removed from the $(n-1)$ survivals, hence, leaving $n-1-R_1$ survival items. Then after the failure of the second item, the sample becomes $n-2-R_1$ in which another sample of size R_2 is randomly selected and removed

* Corresponding author.
E-mail address: a.helu@ju.edu.jo (A. Helu).

http://dx.doi.org/10.1016/j.ress.2014.07.006
0951-8320/

from the remaining survival units. Continue with this process until m failures are observed and all the remaining $n-m-R_1-\cdots R_{m-1}(=R_m)$ surviving units are removed from the experiment. It is assumed that the lifetimes of these n units are independent and identically distributed with common distribution function F. Moreover, n, m and the censoring scheme $R_1, R_2, \ldots, R_m$ are all pre-fixed. Note that if $R_1 = R_2 = \cdots = R_{m-1} = 0$, then $R_m = n-m$ which corresponds to type-II censoring. If $R_1 = R_2 = \cdots = R_m = 0$, then $m=n$ which represents the complete sample.

Many authors have discussed inference under progressive censoring using different lifetime distributions. Among others, Cohen [13], Mann [22], Wingo [33], Balakrishnan and Sandhu [7], Aggarwala and Balakrishnan [1], Balakrishnan and Asgharzadeh [6], Soliman et al. [31]. For a comprehensive recent review of progressive censoring, readers may refer to Balakrishnan [4].

The aim of the article is to consider classical and Bayesian estimation of the unknown parameters of the *IW* distribution under progressively type II censoring. It is observed that the *MLE*s cannot be obtained in a closed form. In this case we suggest to use the approximate *MLE* (*AMLE*). The *AMLE* is obtained by expanding the normal equations using Taylor approximation. On the other hand, while the method of *MLE* is the most popular in terms of theoretical prospective, the least square method (*LSE*) is computationally easier to handle and provides simple closed form solutions for the estimates [15]. We further consider the Bayes estimates of α and β under symmetric and asymmetric loss functions. It is observed that the Bayes estimates cannot be obtained in explicit forms and instead of using numerical techniques, approximation method such as Lindley's approximation is applied.

This article unfolds as follows: In Section 2 we derive the classical methods of estimation, namely *MLE*s, approximate *MLE*s, and *LSE* methods of the unknown parameters. The Bayes method is provided in Section 3. A simulation study is conducted in Section 4. Data analysis and comparison study are in Section 5. All methods that are discussed in this article are illustrated in Section 6 through a real life data example that represents failure times of aircraft windshields. Our conclusion and recommendations are presented in Section 7.

2. Classical estimation procedures

2.1. Maximum likelihood estimator

Suppose that n independent units are placed on a test. The ordered m failures are observed under the type-II progressively censoring scheme $\mathbf{R} = (R_1, \ldots, R_m)$, where $R_i \geq 0$, $n = m + \sum_{i=1}^{m} R_i$. Let $\mathbf{X} = (X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n})$ with $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ denote the progressive type-II right censored data from a population with pdf and cdf given in Eqs. (1) and (2), respectively. For notation simplicity, we will write X_i for $X_{i:m:n}$. The likelihood function based on progressively type-II censored sample (see [5]) is given by

$$L(\alpha,\beta;\mathbf{X}) = c\prod_{i=1}^{m} f(x_i;\alpha,\beta)[1-F(x_i;\alpha,\beta)]^{Ri}, \tag{3}$$

where

$$c = n(n-1-R_1)(n-2-R_1-R_2)\cdots\left(n-\sum_{i=1}^{m}(R_i+1)\right).$$

In accordance with (1), (2) and (3), the log-likelihood function of α and β based on progressive type-II censored sample $\mathbf{X}$ becomes

$$\ln L(\alpha,\beta;\mathbf{X}) = \ln(c) + m\ln(\alpha\beta) - (\beta+1)\sum_{i=1}^{m}\ln(\alpha x_i) - \sum_{i=1}^{m}(\alpha x_i)^{-\beta} + \sum_{i=1}^{m} R_i \ln(1-e^{-(\alpha x_i)^{-\beta}}). \tag{4}$$

The *MLE*s of the parameters α and β can be obtained by deriving (4) with respect to α and β and equating the normal equations to 0 as follows:

$$\frac{\partial \ln L(\alpha,\beta;\mathbf{X})}{\partial\alpha} \propto -m + \sum_{i=1}^{m}(\alpha x_i)^{-\beta} - \sum_{i=1}^{m}\frac{R_i(\alpha x_i)^{-\beta}e^{-(\alpha x_i)^{-\beta}}}{1-e^{-(\alpha x_i)^{-\beta}}} = 0, \tag{5}$$

$$\frac{\partial \ln L(\alpha,\beta;\mathbf{X})}{\partial\beta} = \frac{m}{\beta} - \sum_{i=1}^{m}\ln(\alpha x_i)(1+(\alpha x_i)^{-\beta}) - \sum_{i=1}^{m}\frac{R_i(\alpha x_i)^{-\beta}e^{-(\alpha x_i)^{-\beta}}\ln(\alpha x_i)}{1-e^{-(\alpha x_i)^{-\beta}}} = 0. \tag{6}$$

Notice that there are no explicit solutions to (5) and (6). Hence, numerical methods are applied to solve the required equations.

2.2. Approximate maximum likelihood estimator

Since the *MLE* does not provide explicit estimators for the shape and scale parameters of the *IW* distribution as mentioned before, we derive approximate *MLE* (*AMLE*) for the parameters α and β. Balakrishnan and Vardan [8] develop the *AMLE* procedure. This procedure depends on the Taylor expansion of the likelihood function when the pdf under consideration belongs to the location–scale families. However, the *IW* distribution does not have the location–scale structure required for the *AMLE* procedure, but if we consider the transformation $Y = -\ln X$, then $Y\sim$ Extreme value distribution and this distribution has this feature.

The *pdf* and *cdf* of Y are given respectively by

$$h(y;\mu,\sigma) = \frac{1}{\sigma}e^{(y-\mu)/\sigma} - e^{(y-\mu)/\sigma}, \quad -\infty < y < \infty, \tag{7}$$

and

$$H(y;\mu,\sigma) = 1 - e^{-e^{(y-\mu)/\sigma}}, \tag{8}$$

where $\mu = \ln\alpha$ and $\sigma = 1/\beta$ are the location and scale parameters respectively. Hence, the *AMLE* procedure can be used to estimate the parameters α and β of the *IW* distribution.

Balakrishnan and Aggarwala [5] have calculated the approximate maximum estimators of the Extreme value parameters based on Type-II censoring as follows: Let

$$\alpha_{i:m:n} = 1 - \prod_{j=m-i+1}^{m}\frac{j+\sum_{k=m-j+1}^{m}R_k}{1+j+\sum_{k=m-j+1}^{m}R_k}, \quad i = 1,2,\ldots,m \tag{9}$$

$$\nu_i = \ln(-\ln(1-\alpha_{i:m:n})), \tag{10}$$

$$\gamma_i = e^{\nu_i}(1-\nu_i) \quad \text{and} \quad \beta_i = e^{\nu_i} \geq 0, \tag{11}$$

then the approximate maximum estimators of the Extreme value parameters under progressively Type-II censoring are given by

$$\mu = C\sigma + D \tag{12}$$

$$0 = m\sigma^2 + A\sigma + B \tag{13}$$

where

$$C = \frac{\sum_{i=1}^{m}(R_i+1)\gamma_i - m}{\sum_{i=1}^{m}(R_i+1)\beta_i}, \quad D = \frac{\sum_{i=1}^{m}(R_i+1)\beta_i y_i}{\sum_{i=1}^{m}(R_i+1)\beta_i}$$

$$A = \sum_{i=1}^{m}\left[(R_i+1)\gamma_i - 1\right](y_i - D); \quad B = -\sum_{i=1}^{m}(R_i+1)\beta_i(y_i-D)^2.$$

The solutions to the above equations yield the *AMLE*s

$$\widehat{\sigma}_{AMLE} = \frac{-A+\sqrt{A^2-4mB}}{2m}; \quad \widehat{\mu}_{AMLE} = D + C\widehat{\sigma}_{AMLE}. \tag{14}$$

One of the drawbacks of the *AMLE*s is that they are biased. Moreover, the exact bias of $\widehat{\mu}_{AMLE}$ and $\widehat{\sigma}_{AMLE}$ cannot be theoretically