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Optimal design of accelerated life tests for an extension of the exponential distribution

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ABSTRACT

Accelerated life tests provide information quickly on the lifetime distribution of the products by testing them at higher than usual levels of stress. In this paper, the lifetime of a product at any level of stress is assumed to have an extension of the exponential distribution. This new family has been recently introduced by Nadarajah and Haghighi (2011 [1]); it can be used as an alternative to the gamma, Weibull and exponentiated exponential distributions. The scale parameter of lifetime distribution at constant stress levels is assumed to be a log-linear function of the stress levels and a cumulative exposure model holds. For this model, the maximum likelihood estimates (MLEs) of the parameters, as well as the Fisher information matrix, are derived. The asymptotic variance of the scale parameter at a design stress is adopted as an optimization objective and its expression formula is provided using the maximum likelihood method. A Monte Carlo simulation study is carried out to examine the performance of these methods. The asymptotic confidence intervals for the parameters and hypothesis test for the parameter of interest are constructed.

1. Introduction

Accelerated life testing consists of a variety of test methods for shortening the lifetime of products that are often extremely reliable under normal operating conditions. The aim of such testing is to quickly obtain data which yield desired information on the lifetime of a product under normal use. A simple way to accelerate the lifetime of many products is to run the product at a higher usage stress. The results obtained at the accelerated conditions are analyzed in terms of a model to relate the lifetime to the stress. They are then extrapolated to estimate the lifetime at design conditions. Some key references in the area of the accelerated life testing include Nelson [2], Meeker and Escobar [3], and Bagdonavičius and Nikulin [4]. A special class of the accelerated life testing is *the step-stress accelerated life test*. In this type of testing a unit is first subjected to a specified constant stress x_1 for a specified length of time τ_1. If it does not fail, it is subjected to a higher stress level x_2 for a specified time τ_2, and so on. The stress on a unit is thus increased step by step until it fails. If there is a single change of stress, the accelerated life test is called a simple step-stress test. There is a disadvantage of step-stress tests for reliability estimation. Most products run at constant stress, not step-stress. Thus the model must properly take into account the cumulative effect of exposure at successive stress levels. One such model is *the cumulative exposure model* introduced by Sedyakin [5], Bagdonavičius [6] and Nelson [7]. It assumes that the remaining lifetime of a unit depends only on the current cumulative fraction failed and the current stress, regardless of how the fraction is accumulated. In this paper, we consider the problem of optimal simple step-stress plan for cumulative exposure models under an extension of the exponential distributed lifetime. Optimal plans yield the most accurate estimates of lifetime at the design stress and better results for given test cost and time. The problem of designing *optimum* simple step-stress plans for accelerated life testing has been discussed by Miller and Nelson [8]. Further Bai et al. [9] presented the optimum simple step-stress accelerated life tests for the case where a pre-specified censoring time is involved. The problem of optimally designing simple step-stress accelerated tests in which failed items are replaced with new ones was considered by Bai and Chung [10]. Khamis and Higgins [11] derived the optimum three step step-stress accelerated life testing for Type-I censored data while the optimal step-stress test under progressive Type-I censoring was considered by Gouno et al. [12]. All of these works were based on the assumption of exponentially distributed lifetime. This assumption has limited the application of these results. Despite the computation complexity, several authors have developed similar results mentioned above for the lifetime distributions with non-constant hazard rate. For example, the optimal design of partially accelerated life tests

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under Type-I censoring and optimal simple step-stress plan for cumulative exposure model using *Log-normal* distribution have been studied by Bai et al. [13] and Alhadeed and Yang [14]. Bai and Chun [15] considered the optimal design of life-test sampling plans based on failure-censored accelerated life tests for *Weibull* distribution and an optimal simple step-stress plan for Weibull distribution under Khamis–Higgins model was obtained by Alhadeed and Yang [16]. Srivastava and Shukla [17] discussed an optimum plan for simple step-stress accelerated life tests under the *Log-logistic* model. There are some works on designing accelerated lifetime testing plans without making assumptions about the lifetime distribution of the test units; see Elsayed and Zhang [18] and Haghighi [19]. Also, some studies have developed the Bayesian methods for planning optimal accelerated life testing; see Briš [20], Zhang and Meeker [21], Liu and Tang [22] and Yuan et al. [23].

The aim of this paper is to present a simple step-stress test and drive an optimum plan under an extension of the exponential model, a more flexible one than the exponential model. Recently some generalizations of the exponential distribution are proposed with the hope that they attract wider applicability in reliability, for example exponentiated exponential distribution considered by Gupta and Kundu [24], beta exponential distribution proposed by Nadarajah and Kotz [25] and an extension of the exponential distribution presented by Nadarajah and Haghighi [1]. Our motivations for considering the last new family are as follows. The first motivation is based on its hazard rates. An extension of the exponential distribution does not have the limitations of exponential distribution and provides non-constant hazard rates. It can be used as an alternative for gamma, Weibull and exponentiated exponential distributions. The second motivation is based on the relationship between its probability density function and its hazard function. The new generalization always has a decreasing probability density function and yet allows for increasing, decreasing and constant hazard rates. The gamma, Weibull and exponentiated exponential distributions are widely used in reliability studies. They allow for both monotonically decreasing and unimodal probability density functions. However, they do not allow for an increasing hazard function when their respective probability density functions are monotonically decreasing. This feature could cause some limitations. For such situation the new family is more realistic than of the others. A comprehensive account of the mathematical properties of this family was presented by Nadarajah and Haghighi [1]. An extension of the exponential distribution has a closed form expression for survival function as follows:

$$S(t)=\exp\left\{1-\left(1+\frac{t}{\lambda}\right)^{\alpha}\right\}, \tag{1}$$

for $\alpha>0$, $\lambda>0$ and $t>0$. The corresponding cumulative distribution function (cdf), probability density function (pdf) and the quantile function are

$$F(t)=1-\exp\left\{1-\left(1+\frac{t}{\lambda}\right)^{\alpha}\right\}, \tag{2}$$

$$f(t)=\frac{\alpha}{\lambda}\left(1+\frac{t}{\lambda}\right)^{\alpha-1}\exp\left\{1-\left(1+\frac{t}{\lambda}\right)^{\alpha}\right\}, \tag{3}$$

and

$$Q(p)=\lambda\{(1-\log(1-p))^{1/\alpha}-1\},\quad 0<p<1. \tag{4}$$

Its mean time to failure (MTTF) is defined as

$$E(t)=\lambda\left[-1+e\Gamma\left(1+\frac{1}{\alpha},1\right)\right], \tag{5}$$

where $\Gamma(\cdot,\cdot)$ is the incomplete gamma function.

For $\alpha=1$, the family reduces to the exponential distribution. It can be considered as a special case of the power generalized Weibull distribution proposed by Bagdonavičius and Nikulin [4] and discussed further by Nikulin and Haghighi [26]. It is important to mention that the new distribution can be interpreted as a truncated Weibull distribution, supposing $Z=Y-1/\lambda$, the distribution is the same as that of Z truncated at zero, when Y is a Weibull random variable with shape parameter α and scale parameter λ. The rest of the study is arranged as follows. In Section 2, the simple step-stress model under an extension of the exponential distribution is described. The maximum likelihood estimates, as well as Fisher information matrix, are derived. An optimum design is presented in Section 3. In Section 4, a simulation study is carried out to examine the methods. Optimal stress-change times are obtained. The asymptotic confidence intervals for the parameters and hypothesis test for the parameter of interest are constructed.

2. Estimation for an extension of the exponential step-stress model

The assumptions in our approach are:

(1) Two stress levels x_1 and x_2 $(x_1<x_2)$ are used.
(2) For any level of stress, the lifetime distribution of the test unit follows pdf given by (3).
(3) The scale parameter of cdf $F(t)$ is a log-linear function of stress, i.e.,

$$\log\,\lambda(x_i)=\beta_0+\beta_1x_i,\quad i=0,1,2, \tag{6}$$

where β_0 and $\beta_1(<0)$ are unknown parameters depending on the nature of the product, and the method of test. x_0 is use-stress.
(4) The constant α does not depend on the stress level.
(5) A cumulative exposure model holds.

The test is conducted as follows. All test units are initially placed on low stress x_1, and run until time τ. Then, the stress is changed to high stress x_2, and the test continues until all remaining units fail. We assume that there are units being tested at every stage of stress change. From the assumptions that the model is a cumulative exposure model, and the lifetime distribution is given by (3), the cdf of a test unit under simple step-stress test is

$$F(t)=\begin{cases}F_1(t), & 0\le t<\tau,\\ F_2(t-\tau+\tau'), & \tau\le t<\infty.\end{cases} \tag{7}$$

where τ' is the solution of $F_1(\tau)=F_2(\tau')$. Because $F_i(t)=1-\exp\{1-(1+t/\lambda_i)^{\alpha}\}$, therefore $\tau'=(\lambda_2/\lambda_1)\tau$. Thus the pdf of a test unit is

$$f(t)=\begin{cases}f_1(t), & 0\le t<\tau,\\ f_2\left(t-\tau+\left(\frac{\lambda_2}{\lambda_1}\right)\tau\right), & \tau\le t<\infty.\end{cases} \tag{8}$$

where, for i=1,2,

$$f_i(t)=\frac{\alpha}{\lambda_i}\left(1+\frac{t}{\lambda_i}\right)^{\alpha-1}\exp\left\{1-\left(1+\frac{t}{\lambda_i}\right)^{\alpha}\right\}.$$

Suppose n_i failure times $t_{ij}, j=1,\dots,n_i$, of test units are observed while testing at stress $x_i, i=1,2$. The log-likelihood function for complete data $t_{ij},\ i=1,2, j=1,2,\dots,n_i$, is

$$\log\,L(\beta_0,\beta_1)=n+n\log\,\alpha-n\beta_0-(n_1x_1+n_2x_2)\beta_1 \\ +(\alpha-1)\sum_{j=1}^{n_1}\log(1+e^{-\beta_0-\beta_1x_1}t_{1j})$$