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# Reliability of maintained systems under a semi-Markov setting

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ABSTRACT

A semi-Markov setting is considered in order to study the main dependability measures of a repairable continuous time system under the hypothesis that the evolution in time of its components is described by a continuous time semi-Markov process. Moreover, the main dependability measures of a periodically maintained system are studied. Finally, all the above systems are compared with the corresponding Markov systems where the general repair time distribution is replaced by the exponential distribution with the same mean which is the most commonly used approximation of the original system in practice.


## 1. Introduction

A main engineering problem is the study of system's dependability. Many system's dependability studies are based on Markov processes (see for example [1–4] the references therein and many others), basically due to the fact that a plenitude of known results on Markov processes exist and are also easy to implement. The main disadvantage of the use of Markov processes is that they do not allow any other distribution for the sojourn times beyond the exponential one. A natural generalization of the Markov processes is the semi-Markov processes that allow any distribution for the sojourn times. This is an important generalization since in most real world applications the lifetimes and/or the repair times are not exponentially distributed. In the standard Markov analysis when we have several independent Markov processes, the vector process consisting of them is also a Markov process on the product state space whose generator is given by the direct Kronecker sum of the partial process generator. But this is not the case in the semi-Markov processes. A vector process consisting of semi-Markov processes is not any more a semi-Markov process. This is the main inconvenience of semi-Markov systems.

In this paper our main goal is to study the main dependability measures of existing repairable systems (or subsystems) of which the evolution in time of each component is described by a continuous time semi-Markov process (CTSMP) based on the standard theory of semi-Markov processes [5]. The studied systems can be applied to signaling, telecommunication and several other fields. We consider here that the lifetime and repair times of each component are independent. The lifetime distribution of each component is exponential, while its repair time has a general distribution. In digital electronics, assuming that a constant failure rate is not at all unrealistic because there is generally no wear-out. Each component can be either in the functioning state or in the failure state. Moreover, we would like to compare the studied systems with the corresponding Markov systems where the general repair time distribution is replaced by the exponential distribution with the same mean which is the most commonly used approximation of the original system in practice.

The rest of the paper is organized as follows. In Section 2, a two-component parallel system is presented and studied extensively. In Section 3 we study a two-component periodically maintained system. This dependability study is generalized in a $(n-1)$-out-of-$n$ system. Finally, in Section 4, the studied systems are compared with the corresponding Markov systems. We conclude this paper by providing a short discussion.

## 2. Reliability of semi-Markov systems

In this section we will present and study a two component parallel system. This system appears in several real word applications. For example, it can describe successfully a two channel communication system that needs at least one of the two channels to be operational in order to function properly. If one channel is lost then the system is vulnerable and the loss of the second channel implies loss of service. A stand-by redundancy is not very convenient for several types of industrial applications, such as train control, because, in these architectures, upon failure of the first channel, a switch-over time would be necessary in order to activate the spare channel. So, this implies downtime. If it was

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possible to constantly monitor the main channel and activate a spare (reserve) channel instantly upon failure of the main one, then indeed a cold redundancy would be more "efficient".

The two component parallel systems have been studied by several authors and under different assumptions. Gaver [6,7] studied a two component parallel system assuming that each component has an exponential failure time distribution and a general repair time distribution. Osaki [8], Linton [9] and Subramanian and Ravichandran [10] generalized Gaver's study. Ohashi and Nishida [11] studied Gaver's model in the particular case where both components are identical and all the distributions are general. Variations of Gaver's model have been presented and studied in several papers such as Buzacott [12], Takeda and Osaki [13], Jank [14], Vanderperre [15], Rajamanickam and Chandrasekar [16], and the references therein.

The point here is that the two component parallel systems are studied under a semi-Markov setting. Based on the well-known results on CTSMP, explicit formulas for the main reliability measures are derived.

### 2.1. The system

Let us consider a two component parallel system. The lifetime distribution of each component is an exponential distribution $\varepsilon(\lambda)$ with common failure rate $\lambda$, while its repair time has a general distribution $G(\cdot)$ (Dirac, Lognormal and Weibull are some of the most common distributions employed for the repair time). Moreover, we assume that there is only one repair station and the repaired components are as good as new. Thus, this system can be in one of the following states:

- *State* 0: Both components are operating.
- *State* 1: One component is operating and the repair of the other component has just started.
- *State* 2: The operating component fails while the other component is under repair (the system is not functioning any more).

Let us now assume that $\lambda_0 = 2\lambda$ and $\lambda_1 = \lambda$ are the (constant) failure rates of the two components parallel system when it is in states 0 and 1 respectively. Moreover, let $\{Z_t\}_{t\geq 0}$ be the process that describes the evolution in time of the studied system, with state space $E=\{0,1,2\}$. We will study the process $\{Z_t\}_{t\geq 0}$ up to the first hitting time to the state 2, that is $T_2 = \inf\{t \geq 0 : Z_t = 2\}$. In this case, the process $\{Z_t\}_{0\leq t<T_2}$ is a semi-Markov process, with semi-Markov kernel:

$$\mathbf{Q}(t)=\begin{bmatrix} 0 & Q_{01}(t) & 0 \\ Q_{10}(t) & 0 & Q_{12}(t) \\ 0 & 0 & 0 \end{bmatrix}.$$

In order to specify the kernel of the semi-Markov process, we need to compute the $Q_{ij}(t)$ for $i,j \in E$. For example, $Q_{10}(t)$ is the probability of a transition from state 1 to state 0 up to time $t$. This is true, if the repair of the one component takes place before the failure of the operating one. The repair time is a random variable $T_{10}$ distributed according to the cumulative density function $G(t)$. If a failure occurs earlier than a repair, the system transits from state 1 to state 2. This time is a random variable $T_{12}$ which is exponentially distributed, $\varepsilon(\lambda_1)$. Thus, the probability $Q_{10}(t)$ can be determined as the probability $T_{10} \leq t$ and the random variable $T_{10}$ is less than the variable $T_{12}$. Hence, we have that

$$\begin{aligned} Q_{10}(t) &= \mathrm{P}(T_{10} \leq t \text{ and } T_{12} > T_{10}) \\ &= \int_0^t G(du)\int_u^\infty \lambda_1 e^{-\lambda_1 s}\,ds = \int_0^t G(du) e^{-\lambda_1 u}. \end{aligned}$$

Similarly we can obtain $Q_{12}(t)$. Thus, we have that

$$Q_{01}(t) = 1-\exp\{-\lambda_0\ t\}, \quad Q_{10}(t) = \int_0^t \exp\{-\lambda_1\ u\}G(du),$$

$$Q_{12}(t) = \int_0^t \lambda_1 \exp\{-\lambda_1\ u\}\overline{G}(u)\,du.$$

The transition kernel of the embedded Markov chain $(J_n)_{n\geq 0}$ is

$$\mathbf{P}=\begin{bmatrix} 0 & 1 & 0 \\ p & 0 & 1-p \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where } p=\int_0^\infty \exp(-\lambda_1\ u)G(du).$$

The mean sojourn time $m_i$ in state $i$ is $m_i = \int_0^\infty tH_i(dt) = \int_0^\infty (1-H_i(t))\,dt$, where $H_i(t) = \sum_{j\in E} Q_{ij}(t)$ is the cumulative distribution function of the sojourn time in state $i$. In this case, we have $m_0 = 1/\lambda_0$, and $m_1 = \int_0^\infty (1-H_1(t))\,dt$, where $H_1(t) = Q_{10}(t)+Q_{12}(t)$.

In order to compute the main dependability measures, we will divide the states of the system into two subsets, the functioning and failed ones. Denote by $U$ the operational states of the system (the up states) and by $D$ the subsets of failure states (the down states), i.e. $E = U \cup D$, with $U \cap D = \varnothing$, $U \neq \varnothing$ and $D \neq \varnothing$. It is clear in this case that $U=\{0,1\}$ and $D=\{2\}$. We consider now the natural matrix partition of the matrices $\mathbf{Q}(t)$ and $\mathbf{P}$ corresponding to the state space partitions $U$ and $D$. Thus, we have

$$\mathbf{Q}(t)=\begin{bmatrix} \mathbf{Q}_{11}(t) & \mathbf{Q}_{12}(t) \\ \mathbf{Q}_{21}(t) & \mathbf{Q}_{22}(t) \end{bmatrix} \quad \text{and} \quad \mathbf{P}=\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix},$$

where $\mathbf{Q}_{11}(t)$ and $\mathbf{P}_{11}$ are the restriction of the semi-Markov kernel and of the transition kernel of the EMC into the subset $U$, respectively. Let us also assume that the system starts to work in the perfect state. Thus, the initial distribution is $a=(1,0,0)$ and the truncation of it in $U$ is $a_1 = (1,0)$.

Let us assume that the system starts to work at time $t=0$ and the event $\{Z_t = i,\ i \in U\}$ means that the system is in the operating mode $i$ at time $t$. The reliability of the system at time $t$ is defined as the probability that the system has been functioning without failure in the interval $[0,\ t]$, that is

$$R(t) = \mathrm{P}(Z_s \in U, \forall s \in [0,t]).$$

If we define the conditional reliability by

$$R_i(t) = \mathrm{P}(Z_s \in U, \forall s \in [0,t] | Z_0 = i), \quad i \in U,$$

then for any initial distribution $a$, we have $R(t) = \sum_{i\in U} a_i R_i(t)$. It is obvious that $R_j(t) = 0$ for $t \geq 0$, if $j \in D$. The conditional reliability $R_i(t)$ satisfies the following Markov Renewal Equation (MRE):

$$R_i(t) = 1-H_i(t)+\sum_{j\in U}\int_0^t Q_{ij}(ds)R_j(t-s). \tag{1}$$

Eq. (1) can be approximated by

$$R_i(t) \approx 1-H_i(t)+\sum_{j\in U}\sum_{\ell=1}^{k} R_j(t-x_\ell)(Q_{ij}(x_\ell)-Q_{ij}(x_{\ell-1})),$$

where $0=x_0<x_1<\cdots<x_k=t$, which can be used recursively in order to obtain $R_i(t)$, starting from an initial point $R_i(0)=1$ (see [5]). Moreover, Eq. (1) can be written in matrix form as follows:

$$\mathbf{R}(t) = (\mathbf{I}-\mathbf{H}_1)(t)1_{s_1} + \mathbf{Q}_{11} * \mathbf{R}(t) \tag{2}$$

where $1_{s_1}$ is a $s_1$-column vector whose elements are all equal to unity. Solving the MRE (2), we get that the reliability of the system at time $t$, starting at time 0 in a functioning state, is

$$\mathbf{R}(t) = a_1(\mathbf{I}-\mathbf{Q}_{11})^{(-1)} * (\mathbf{I}-\mathbf{H}_1)(t)1_{s_1}$$

where the matrix convolution product is denoted by $*$, $\mathbf{I}$ is the identity element for the matrix convolution product, i.e. $\mathbf{I}*A = A*\mathbf{I} = A$ and $A^{(-1)}$ is the inverse of $A$ with respect to convolution. One way to compute the matrix $(1-\mathbf{Q}_{11}(t))^{(-1)}$ is by explicit inversion within convolution algebra. This is obtained by the usual