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On selection of optimal stochastic model for accelerated life testing

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ABSTRACT

This paper deals with the problem of proper lifetime model selection in the context of statistical reliability analysis. Namely, we consider regression models describing the dependence of failure intensities on a covariate, for instance, a stressor. Testing the model fit is standardly based on the so-called martingale residuals. Their analysis has already been studied by many authors. Nevertheless, the Bayes approach to the problem, in spite of its advantages, is just developing. We shall present the Bayes procedure of estimation in several semi-parametric regression models of failure intensity. Then, our main concern is the Bayes construction of residual processes and goodness-of-fit tests based on them. The method is illustrated with both artificial and real-data examples.

1. Introduction

Accelerated life testing is a standard approach to gather information on the survival time of highly reliable devices. One of the goals of statistical analysis consists in the construction of a model of the time to failure dependence on the 'stressor' (in a quite wide sense). As a rule, the stressor is taken as a covariate in a regression model of the lifetime. The model should be selected in such a way that the information obtained under the over-stress could be extrapolated to standard stress conditions. These problems, including the test design, selection of models, procedures of statistical analysis, have been treated in a number of papers and books, for instance [9,10,4,5]. Nowadays, many authors prefer the Bayes approach, though mostly in the framework of parametrized (e.g. Weibull) models. Simultaneously, computations are supported by the Markov Chain Monte Carlo (MCMC) generation of posterior and predictive distribution, as in Van Dorp and Mazzuchi [14]. In the same context, Erto and Giorgio [6] accent the advantage of utilization of prior information, an experience from past tests as well as the expert knowledge. Wang et al. [16] model and analyze the process of degradation, instead of failure times directly, using a Gauss or gamma process as a baseline source of uncertainty. They provide the Bayes method and the MCMC procedure enabling one to combine accelerated laboratory tests with field data in order to analyze the reliability of system.

The selection of a proper stochastic model is just one of the steps of statistical analysis. The model criticism, including the goodness-of-fit tests, should follow. Therefore, the methods of goodness-of-fit statistical testing are in the center of our attention. In the present paper we consider three basic semi-parametric regression models describing the dependence of intensity of failures on covariates, in our context on the load, stress or other conditions of usage.

In the framework of intensity models for lifetime data, the goodness-of-fit tests are often based on the analysis of residual process (martingale residuals). The residual process is defined as a difference between estimated cumulated intensity and observed counting process of failures (see for instance [3]). Hence, the residual process is constructed from the observed data, its properties depend on the properties of the estimator of the cumulated hazard rate. In a case without regression, as well as in Aalen's additive regression model, residual processes are the martingales [15]. In some other cases, as is Cox's model or the accelerated failure time (AFT) model, the behavior of estimates, and therefore of residuals, is more complicated. That is why the tests are often performed just graphically [1]. Approximate critical regions for tests can also be obtained by random generation from asymptotic distribution of residual processes. Relevant theoretical results can be found for instance in Andersen et al. [3], Lin et al. [11], and Bagdonavicius and Nikulin [4]. In such cases, the Bayes approach can offer a reasonable alternative, especially when connected with the MCMC methods (an overview of the MCMC is given for instance in [7]). The present paper deals prevailingly with semi-parametric intensity models consisting of a parametric regression part and a nonparametric baseline hazard rate. For the Bayes solution, its representation can be made from piecewise-constant functions (or from splines or from other functional basis), in the way used in Arjas and Gasbarra [2]. Once a posterior sample of

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hazard rate (i.e. representation of its posterior distribution obtained by the MCMC procedure) is available, we can construct a sample representing cumulated intensities and corresponding residuals.

Let us here also recall another approach to the Bayes analysis in the AFT model. It utilizes logarithmic model formulation. Instead of the baseline hazard rate of a baseline survival time T_0 it deals with the density for $\log T_0$. Often, its prior is constructed as a mixture of the Gauss densities with weights given by Dirichlet distributions (as for instance in [8]). However, complications are caused by censoring and have to be overcome by an additional generation of would-be non-censored values, i.e. by a data augmentation. It is actually a randomized version of the EM algorithm.

The present paper has the following structure: In the next section, the notion of martingale residuals is recalled, then the Bayes nonparametric approach to intensity modeling is described. While these sections are more-less introductory, the core of the paper lies in Sections 4–6 dealing with regression models, methods of analysis and their Bayesian counterparts. Utilization of the MCMC procedures leads to the Bayes 'empirical' construction of residual processes. The method is finally, in Section 7, illustrated with both artificial and real-data examples.

2. Martingale residuals

In order to introduce the notion of martingale residuals, we shall first consider a standard survival data case, without any dependence on covariates. Let us imagine that a set of i.i.d. random variables T_i, survival times of n objects of the same type, is observed. Alternatively, we may consider their counting processes $N_i(t)$, each having maximally 1 count (at the time of failure, T_i), or being censored without failure. Further, let us also consider indicator processes (of being at risk) $Y_i(t)$, $Y_i(t)=0$ after failure or censoring, $Y_i(t)=1$ otherwise. As the lifetimes are i.i.d., corresponding counting processes have the same common hazard rate $h(t)\ge 0$. The cumulated hazard rate is then $H(t)=\int_0^t h(s)\,ds$. It follows that the intensity of $N_i(t)$ is $a_i(t)=h(t)\cdot Y_i(t)$. Notice a difference between those two notions: the hazard rate is a characteristic of distribution, namely here $h(t)=-d(\ln \overline{F})(t)/dt$, where $\overline{F}(t)=1-F(t)$ is a survival function, complement to the distribution function, while the intensity depends on realizations of processes $Y_i(t)$. It is assumed that the data are observed on a finite time interval $t\in[0,T]$, $N_i(0)=0$.

Let us also define sums of individual characteristics, namely counting process $N(t)=\sum_{i=1}^n N_i(t)$ counting number of failures, further $Y(t)=\sum_{i=1}^n Y_i(t)$, cumulated intensities $A_i(t)=\int_0^t a_i(s)\,ds$ and $A(t)=\sum_{i=1}^n A_i(t)$, so that here $A(t)=\int_0^t h(s)Y(s)\,ds$.

In theoretical studies on lifetime models, many results are based on martingale—compensator decomposition of counting process, namely that $N_i(t)=A_i(t)+M_i(t)$, so that also $N(t)=A(t)+M(t)$, where $M_i(t)$ and $M(t)$ are martingales with zero means, conditional variance processes (conditioned by corresponding filtration, a nondecreasing set of σ-algebras $\mathcal{F}(t^-)$) are $\langle M_i\rangle(t)=A_i(t)$ and $\langle M\rangle(t)=A(t)$. Naturally, martingales have non-correlated increments, and $M_i(t)$ are also non-correlated mutually (for different i).

Then it is quite reasonable to consider a residual process (martingale residuals)

$$R(t)=N(t)-\hat{A}(t)=M(t)+A(t)-\hat{A}(t)$$

as a tool for testing model fit. Here $\hat{A}(t)$ is the estimated cumulated intensity. Hence, the residual process is constructed from the observed data, and its properties depend mainly on the properties of the estimator of the cumulated hazard rate, because $\hat{A}(t)=\int_0^t Y(s)\,d\hat{H}(s)$. Tests are then performed either graphically or numerically, critical borders for assessing the goodness-of-fit are based on the asymptotic properties of estimates.

2.1. Properties of residuals

The most common estimator of cumulated hazard rate $H(t)$ is the Nelson–Aalen estimator, which has the form

$$\hat{H}(t)=\int_0^t \sum_{i=1}^n \frac{dN_i(s)}{\sum_{j=1}^n Y_j(s)}=\int_0^t \frac{dN(s)}{Y(s)},$$

so that it is a piecewise constant function with jumps $d\hat{H}(s)=dN(s)/Y(s)$ at times where failures have occurred. Its asymptotic properties, namely uniform on $[0,T]$ consistency in probability and asymptotic normality when $n\to\infty$, are well known (for review of survival analysis, see for instance [10]). More precisely, the following convergence in distribution on $[0,T]$ to the Brown motion process $\mathcal{B}$ holds

$$\sqrt{n}(\hat{H}(t)-H(t))\to^d \mathcal{B}(V(t)),\quad V(t)=\int_0^t \frac{h(s)\,ds}{c_0(s)},$$

where we assume the existence of $c_0(s)=P-\lim Y(s)/n$, uniformly in $[0,T]$, $c_0(s)\ge\varepsilon>0$. Hence, it is possible to construct Kolmogorov–Smirnov type confidence bands for $H(t)$ as well as pointwise confidence intervals. Again, a consistent, uniformly in $[0,T]$, estimator of $V(t)$ is available: $\hat{V}(t)=\int_0^t n\,dN(s)/Y(s)^2$.

In the present contribution we are interested mainly in the properties of residual process $R(t)=N(t)-\hat{A}(t)$. Notice that here $\hat{A}(t)=N(t)$ directly, so that it is preferred to construct residuals in data subsets (strata), $S\subset\{1,..,n\}$. Thus, let us define

$$R_S(t)=N_S(t)-\hat{A}_S(t)=M_S(t)+A_S(t)-\hat{A}_S(t),$$

where we denote again $N(t)=\sum_{i=1}^n N_i(t)$, $N_S(t)=\sum_{i\in S}N_i(t)$, similarly for $Y(t), M(t), A(t), \hat{A}(t)$. As

$$\hat{A}_S(t)=\int_0^t \sum_{i\in S} d\hat{H}(r)Y_i(r)=\int_0^t \frac{dN(r)}{Y(r)}\cdot Y_S(r)$$
$$=\int_0^t \frac{dH(r)Y(r)+dM(r)}{Y(r)}\cdot Y_S(r)=A_S(t)+\int_0^t \frac{dM(r)}{Y(r)}\cdot Y_S(r),$$

we obtain that (with notation $\overline{S}$ – complement of S)

$$R_S(t)=M_S(t)-\int_0^t \frac{dM(r)}{Y(r)}\cdot Y_S(r)=\int_0^t \frac{dM_S(r)Y_{\overline{S}}(r)-dM_{\overline{S}}(r)Y_S(r)}{Y(r)}.$$

From its structure it follows that the process $R_S(t)$ has non-correlated increments, conditioned variance (by σ-algebras $\mathcal{F}(t^-)$) of $(1/\sqrt{n})\,dR_S(t)$ is

$$\frac{dH(t)}{nY(t)^2}(Y_{\overline{S}}(t)Y_S(t)^2+Y_{\overline{S}}(t)^2Y_S(t))\sim dH(t)\frac{c_S(t)c_{\overline{S}}(t)}{c_0(t)},$$

where we again assume that there exist P-limits $Y_S(t)/n\to c_S(t)$, $Y_{\overline{S}}(t)/n\to c_{\overline{S}}(t)$, $Y(t)/n\to c_0(t)$, uniform in $t\in[0,T]$, bounded away from zero. Then $(1/\sqrt{n})R_S(t)\to^d\mathcal{B}(V_R(t))$, i.e. it converges to the Brown motion process, too, and the asymptotic variance function $V_R(t)$ is consistently estimable by

$$\hat{V}_R(t)=\int_0^t \frac{d\hat{H}(r)Y_S(r)Y_{\overline{S}}(r)}{nY(r)}=\int_0^t \frac{dN(r)Y_S(r)Y_{\overline{S}}(r)}{nY(r)^2}.$$

Hence, if assumptions of our model hold, the process

$$\frac{1}{\sqrt{n}}\frac{R_S(t)}{(1+\hat{V}_R(t))}$$

should behave asymptotically as the Brown bridge process. It can be tested by the Kolmogorov–Smirnov criterion (or other similar criteria, as is the Cramer–von Mises test). Therefore, in such a simple case of survival model without any non-heterogeneity, the method can be used for assessing the model fit in different