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General nodal expansion method for multi-dimensional steady and transient convection–diffusion equation

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ABSTRACT

A general nodal expansion method (GNEM) is developed to solve the multi-dimensional steady and transient convection–diffusion equation in this paper. The developed GNEM is an integration of the modified nodal integral method (MNIM) and nodal expansion method (NEM). Firstly, the local analytical solution of the transverse integral equation from MNIM is obtained under the framework of the basis function expansion method in NEM. Secondly, as to GNEM for time-dependent problems, the full space–time nodal method is adopted. Thirdly, GNEM borrows the nodal balance equation from NEM to ensure that the uniqueness of nodal-average variables and the conservation of all the pseudo-sources in each direction are automatically satisfied, which saves the GNEM from the complex derivation and calculations of the nodal-average variables and the zeroth order pseudo-source terms in MNIM. In this paper, GNEM has integrated the multi-dimensional transient and steady problems into the unified discrete formulation. Besides, the code based on GNEM can be easily developed from the existing NEM code since GNEM-base models have a unified formulation with conventional NEM—they only differ in the two control parameters. Meanwhile, this kind of unified formulation also makes it easier to select GNEM and conventional NEM in different computational domains for a test problem. Finally GNEM is formally developed and numerical results show that GNEM has an accuracy consistent with the MNIM, or even higher accuracy than MNIM.

1. Introduction

Nodal methods with transverse integration technique have been widely adopted in reactor physics analysis due to its high efficiency and accuracy (Lawrence, 1986). In general, the basic idea is that the multi-dimensional partial differential equation is reduced into multiple one-dimensional transverse integral equations by the transverse integration procedure, and then the one-dimensional transverse integral equation can be solved by different methods, which lead to different kinds of transverse-integrated nodal methods, and among these methods, there are two popular methods, nodal expansion method (NEM) and nodal integral method (NIM) (Lee, 2001).

In terms of the advantages of transverse-integrated nodal methods, some nodal methods are successfully extended to solve the thermal hydraulic problems, like NIM and later developed modified NIM (MNIM). Both of them have already been developed to solve steady and transient convection diffusion equations, Burgers equations and Navier–Stokes equations (Wilson et al., 1988; Esser and Witt, 1993; Michael, 1995; Michael and Dorning, 2001**b**; Wang, 2005; Singh, 2008; Uddin, 1997). For steady convection diffusion equations, MNIM has been proved to be more accurate and more computationally efficient than high-order numerical schemes of finite difference method (FDM) or finite volume method (FVM), like the local exact consistent upwind scheme of second order (LECUSSO) (Michael and Dorning, 2001b; Gunther, 1992). For transient problems, there are usually two ideas for MNIM. One idea is that the temporal and spatial operators are both discretized by nodal methods, and the other is that temporal operator is firstly discretized by using the lower order backward FDM, and then the spatial operator and the terms resulting from time discretization are discretized using the nodal methods. Numerical properties of space and time for the two ideas are studied in the previous research (Uddin, 1997; Zhou and Li, 2014a; Michael and Dorning, 2001a).

Recently, convectional NEM has been extended to solve multi-dimensional, steady convection–diffusion equations. The numerical properties, including accuracy, stability and numerical diffu-

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sion, are successfully studied in detail (Zhou and Li, 2014b; Zhou et al., 2015). It is proved that the conventional NEM is superior to second order upwind scheme and (quadratic interpolation for convective kinematics) QUICK scheme. Also, nodal integral expansion method (NIEM) has been developed. However, this approach can be only applied to solve one-dimensional, transient convection–diffusion equation (Lee, 2011). In addition, a modified nodal expansion method (MNEM) has also been presented to solve multi-dimensional steady and transient convection–diffusion equation (Deng, 2013). The key ideas lie in that an exponential function, coming from the analytical solution of the one-dimensional convection diffusion equation with constant physical parameters, is introduced into the series of basis functions. Then the temporal operator is treated using FDM. However, due to the fact that the introduction of the above exponential function does not take into account the remaining time-discrete terms, it will be likely to cause greater numerical errors for MNEM.

Therefore, a general nodal expansion method is developed to solve multi-dimensional steady and transient convection–diffusion equation in this paper by making full use of the respective features of modified nodal integral method (MNIM) and conventional nodal expansion method (NEM). From the MNIM, it takes advantage of the ideas that the transverse integral equations can be analytically solved at the each local mesh and that the time variable is treated using nodal methods. From the NEM, nodal balance equation and the framework of the basis function expansion method are adopted so that the local analytical solutions of transverse integral equations are obtained by the basis function expansion method. The detailed derivation process and features of GNEM are presented in Section 2. Some numerical experiments are carried out to test the accuracy and efficiency of GNEM in Section 3, with a comparison of those of conventional NEM and MNIM. Section 4 gives a brief summary and discussion.

2. GNEM's formalism for steady and transient problems

The three-dimensional convection–diffusion equation in Cartesian geometry is:

$$\beta\left(\frac{\partial\phi}{\partial t}\right)+U\frac{\partial\phi}{\partial x}+V\frac{\partial\phi}{\partial y}+W\frac{\partial\phi}{\partial z}-\Gamma\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right)=Q \quad (1)$$

where $\beta=0$ for steady problems and $\beta=1$ for transient problems. U, V and W are the respective velocity component in the different coordinate direction. Γ is diffusion coefficient, and Q is the source term. Different ϕ, Γ and Q can represent different transport equations. For example, when ϕ represents velocity, Γ represents viscosity and Q represents pressure gradient term, then Eq. (1) represents Navier–Stokes equation; when ϕ represents temperature, Γ represents thermal conductivity, Eq. (1) represents energy conservation equation. Therefore Eq. (1) is chosen as the research object and after that, GNEM can be easily extended to solve other types of problems. The domain is divided into $I\times J\times K\times M$ nodes and the size of each node is $[-h_x^{i,j,k},h_x^{i,j,k}]\times[-h_y^{i,j,k},h_y^{i,j,k}]\times[-h_z^{i,j,k},h_z^{i,j,k}]\times[-\tau_m,\tau_m]$, in which i, j, k and m are the nodes' indexes in the x, y, z and t direction respectively, with $i=1,\ldots,I$, $j=1,\ldots,J$, $k=1,\ldots,K$, $m=1,\ldots,M$.

For GNEM, temporal and spatial operator is both discretized by nodal methods, that is, the temporal and spatial operators are both treated by the transverse integration process. Eq. (1) is firstly reduced into four one-dimensional transverse-integrated equations in the x, y, z and t directions by transverse integration strategy. Then each transverse-integrated equation within the time–space nodal (i,j,k,m) is analytically solved by the special basis function expansion method, and these expansion coefficients are determined in the same way as the convectional NEM. After that, by imposing continuity of transverse-integrated variables and diffusion currents at the node interface, a set of coupled discrete equations are obtained in terms of nodal average variables and expansion coefficients of true source term, where the nodal average variables can easily be solved by the nodal balance equation. The derivations are discussed in detail below.

2.1. Transverse integration process

By applying the transverse integration strategy in the time–space nodal (i,j,k,m), transverse-integrated equations in the x, y and z directions are yielded by, respectively:

$$U\frac{d\phi_x(x)}{dx}-\Gamma\frac{d^2\phi_x(x)}{dx^2}=S_x(x)=Q_x(x)-L_x(x) \quad (2)$$

$$V\frac{d\phi_y(y)}{dy}-\Gamma\frac{d^2\phi_y(y)}{dy^2}=S_y(y)=Q_y(y)-L_y(y) \quad (3)$$

$$W\frac{d\phi_z(z)}{dz}-\Gamma\frac{d^2\phi_z(z)}{dz^2}=S_z(z)=Q_z(z)-L_z(z) \quad (4)$$

that is

$$F_r\frac{d\phi_r(r)}{dr}+\frac{dJ_r(r)}{dr}=S_r(r)=Q_r(r)-L_r(r)$$
$$J_r(r)=-\Gamma\frac{d\phi_r(r)}{dr} \quad (5)$$

where the indexes (i,j,k,m) are omitted for simplify on all the above variables and they will be added later. U, V, W and Γ are the average values in the time–space nodal (i,j,k,m); $r=x,y,z$; $F_x=U$, $F_y=V$, $F_z=W$. $\phi_r(r)$ is the r-dependent transverse-integrated variable. The pseudo-source terms $S_r(r)$ are divided into two terms: transverse-integrated true source term $Q_r(r)$ and transverse leakage terms $L_r(r)$. For steady problems $\beta=0$, $\phi_r(r)$, $Q_r(r)$ and $L_r(r)$ are defined as:

$$\phi_r(r)=\frac{1}{4h_\xi h_\eta}\int_{-h_\xi}^{h_\xi}\int_{-h_\eta}^{h_\eta}\phi(r,\xi,\eta)d\xi\,d\eta \quad (6)$$

$$Q_r(r)=\frac{1}{4h_\xi h_\eta}\int_{-h_\xi}^{h_\xi}\int_{-h_\eta}^{h_\eta}Q(r,\xi,\eta)d\xi\,d\eta \quad (7)$$

$$L_r(r)=\frac{1}{4h_\xi h_\eta}\int_{-h_\xi}^{h_\xi}\int_{-h_\eta}^{h_\eta}\left(F_\xi\frac{\partial\phi}{\partial\xi}-\Gamma\frac{\partial^2\phi}{\partial\xi^2}+F_\eta\frac{\partial\phi}{\partial\eta}-\Gamma\frac{\partial^2\phi}{\partial\eta^2}\right)d\xi\,d\eta \quad (8)$$

where $r=x,\ y,\ z\neq\xi\neq\eta$; $\xi=y,\ z,\ x$; $\eta=z,\ x,\ y$. For transient problems, $\beta=1$, and extra time integrations are needed for the definition of $\phi_r(r)$, $Q_r(r)$ and $L_r(r)$.

$$\phi_r(r)=\frac{1}{2\tau}\cdot\frac{1}{4h_\xi h_\eta}\int_{-\tau}^{\tau}\int_{-h_\xi}^{h_\xi}\int_{-h_\eta}^{h_\eta}\phi(r,\xi,\eta,t)d\xi\,d\eta\cdot dt \quad (9)$$

$$Q_r(r)=\frac{1}{2\tau}\cdot\frac{1}{4h_\xi h_\eta}\int_{-\tau}^{\tau}\int_{-h_\xi}^{h_\xi}\int_{-h_\eta}^{h_\eta}Q(r,\xi,\eta,t)d\xi\,d\eta\cdot dt \quad (10)$$

$$L_r(r)=\frac{1}{2\tau}\cdot\frac{1}{4h_\xi h_\eta}\int_{-\tau}^{\tau}\int_{-h_\xi}^{h_\xi}\times\int_{-h_\eta}^{h_\eta}\left(F_\xi\frac{\partial\phi}{\partial\xi}-\Gamma\frac{\partial^2\phi}{\partial\xi^2}+F_\eta\frac{\partial\phi}{\partial\eta}-\Gamma\frac{\partial^2\phi}{\partial\eta^2}\right)d\xi\,d\eta\cdot dt \quad (11)$$

In addition, the transverse-integrated equation in the time t direction can be obtained by transverse integration strategy.

$$\frac{d\phi_t(t)}{dt}=S_t(t)=Q_t(t)-L_t(t) \quad (12)$$