Annals of Nuclear Energy 81 (2015) 342–353

Contents lists available at ScienceDirect

Annals of Nuclear Energy

journal homepage: www.elsevier.com/locate/anucene

Relationship of the SPN_A_N method to the even-parity transport equation

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ARTICLE INFO

Article history:
Received 20 December 2014
Received in revised form 14 March 2015
Accepted 16 March 2015
Available online 4 April 2015

Keywords:
SPN
Fourier transform
Even-parity equation
Neutron transport

ABSTRACT

There are two distinct aspects to this paper. Firstly, it is shown that the A_N form of the SPN equations can be derived from a suitably discretised form of the even parity transport equation. The procedure for doing this is explained and some limitations of SPN_A_N theory are illustrated. Secondly, to show the magnitude of the errors in A_N theory we have taken the problem of an infinitely repeating lattice in which the cell structure is symmetric but can contain any number of sub-regions. The regions can be elliptical and/or rectangular. Because of the repeating lattice, we can express the flux in the cell in the form of a Fourier series. The expansion coefficients of the series may then be obtained from a set of linear equations. It is found that the series converges very slowly and a large number of terms is required which can significantly increase the computational time required for accurate solutions. Use is made of a high order SN ($N = 200$) method to compare and test convergence of the Fourier method for an exact case and for the SPN equivalent.

1. Introduction

There still appear to be some questions regarding the origin, the accuracy and the validity of the SPN equations (Gelbard, 1960). In this note we wish to comment on the similarities between the A_N method of Coppa and Ravetto (1982), Ciolini et al. (2002, 2006) and the even-parity transport equation (Ackroyd, 1997). In addition, we will extend our Fourier transform technique (Wood and Williams, 1972; Hall et al., 2012) to the solution of the even-parity equation and the associated A_N equations for an infinite, repeating lattice. In general, we consider a rectangular cell with an ellipsoidal fuel rod cross section; the major part of the numerical calculations will, however, consider a square cell with a circular cross section rod and, in one case, avoided gap between moderator and fuel. As far as the A_N equations are concerned, we will show that they are a special type of approximation to the even parity equation and are exact for the case of a spatially constant total cross section in the cell but fail when the cross section varies with position in the cell. A detailed description of SPN and A_N equations is given by McClarren (2010) and others in the same volume of Transport Theory and Statistical Physics. Our paper is offered as a further contribution to this fascinating subject.

The numerical work involved, concerns the summation of a double trigonometric series which converges slowly. When the Fourier method for solving this type of problem was first proposed by one of the authors in 1972, computing facilities were as nothing compared with those of today. For this reason we can now sum the slowly converging series to a reasonable degree of accuracy and this enables benchmarks to be established for a number of important cell properties, e.g. the disadvantage factor and associated average fluxes. We illustrate the work by calculating flux profiles and average fluxes in two and three region cells with and without void regions and point out the limitations of the SPN_A_N method. A detailed explanation of the numerical procedure for evaluating the series is given.

2. General theory

The one speed, even-parity transport equation may be written for isotropic scattering as (Ackroyd, 1997)

$$-\boldsymbol{\Omega}\cdot\nabla\left[\frac{1}{\Sigma(\mathbf{r})}\boldsymbol{\Omega}\cdot\nabla\phi^{+}(\mathbf{r},\boldsymbol{\Omega})\right]+\Sigma(\mathbf{r})\phi^{+}(\mathbf{r},\boldsymbol{\Omega}) = \frac{1}{4\pi}\Sigma_s(\mathbf{r})\phi_0(\mathbf{r})+\frac{1}{4\pi}Q(\mathbf{r}) \quad (1)$$

where the even-parity flux is:

$$\phi^{+}(r,\boldsymbol{\Omega}) = \frac{1}{2}(\phi(r,\boldsymbol{\Omega})+\phi(r,-\boldsymbol{\Omega})) \quad (2)$$

$\phi(\mathbf{r},\boldsymbol{\Omega})$ being the normal angular flux. For rectangular co-ordinates:

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http://dx.doi.org/10.1016/j.anucene.2015.03.014

$$\mathbf{\Omega}\cdot\nabla = \sqrt{1-\mu^2}\left[\cos\psi\frac{\partial}{\partial x}+\sin\psi\frac{\partial}{\partial y}\right]+\mu\frac{\partial}{\partial z}$$

and for our two-dimensional cell the axial flux is flat and so the last term is absent in the calculations that follow.

Now the A_N equations (a particular form of the SPN equations) can be written as Ciolini et al. (2002):

$$\mu_a^2\nabla\cdot\left(\frac{1}{\Sigma(\mathbf{r})}\nabla\varphi_a(\mathbf{r})\right)-\Sigma(\mathbf{r})\varphi_a(\mathbf{r})+\Sigma_s(\mathbf{r})\sum_{\bar a=1}^{N}w_{\bar a}\varphi_{\bar a}(\mathbf{r})+S(\mathbf{r}) = 0 \quad (a=1,2,\ldots,N) \tag{3}$$

where w_a are the Gauss–Legendre weights and μ_a the roots of $P_{2N-1}(\mu)=0$. It is stressed that the μ_a do not have any physical meaning, except in the one-dimensional case where they correspond to the cosine of the neutron's direction (Ravetto, 2014). Also the total scalar flux is:

$$\phi_0(\mathbf{r}) = \sum_{a=1}^{N}w_a\varphi_a(\mathbf{r}) \tag{4}$$

We now wish to compare Eq. (3) with the even-parity equation. If we set:

$$\mathbf{J}_a(\mathbf{r}) = -\frac{\mu_a^2}{\Sigma(\mathbf{r})}\nabla\varphi_a(\mathbf{r}) \tag{5}$$

then we have from (3) and (5):

$$-\nabla\cdot\mathbf{J}_a(\mathbf{r})-\Sigma(\mathbf{r})\varphi_a(\mathbf{r})+\Sigma_s(\mathbf{r})\sum_{\bar a=1}^{N}w_{\bar a}\varphi_{\bar a}(\mathbf{r})+S(\mathbf{r})=0 \tag{6}$$

and:

$$\Sigma(\mathbf{r})\mathbf{J}_a(\mathbf{r}) = -\mu_a^2\nabla\varphi_a(\mathbf{r}) \tag{7}$$

Multiplying Eq. (6) by w_a and summing over a, leads to:

$$-\nabla\cdot\mathbf{J}(\mathbf{r})-\Sigma_{abs}(\mathbf{r})\phi_0(\mathbf{r})+S(\mathbf{r})=0 \tag{8}$$

where:

$$\mathbf{J}(\mathbf{r}) = \sum_{a=1}^{N}w_a\mathbf{J}_a(\mathbf{r}) \tag{9}$$

also multiplying Eq. (7) by w_a and summing over a, we have:

$$\Sigma(\mathbf{r})\mathbf{J}(\mathbf{r}) = -\sum_{a=1}^{N}w_a\mu_a^2\nabla\varphi_a(\mathbf{r}) \tag{10}$$

If we now integrate the even-parity equation (1) over $\mathbf{\Omega}$, we find:

$$-\left(\frac{\partial}{\partial x}\frac{1}{\Sigma(\mathbf{r})}\frac{\partial K_x(\mathbf{r})}{\partial x}+\frac{\partial}{\partial y}\frac{1}{\Sigma(\mathbf{r})}\frac{\partial K_y(\mathbf{r})}{\partial y}+\frac{\partial}{\partial z}\frac{1}{\Sigma(\mathbf{r})}\frac{\partial K_z(\mathbf{r})}{\partial z}\right)+\Sigma_{abs}(\mathbf{r})\phi_0(\mathbf{r})=Q(\mathbf{r}) \tag{11}$$

where:

$$K_\gamma(\mathbf{r}) = \int d\mathbf{\Omega}\Omega_\gamma^2\phi(\mathbf{r},\ \mathbf{\Omega}),\ \gamma=x,y,z \tag{12a}$$

and we have used (Ackroyd, 1997, page 248):

$$\int d\mathbf{\Omega}\Omega_\alpha\Omega_\beta\phi^+(\mathbf{r},\mathbf{\Omega})=0,\quad \alpha\neq\beta \tag{12b}$$

Clearly therefore we may write the current $\mathbf{J}$ as:

$$J_x(\mathbf{r}) = -\frac{1}{\Sigma(\mathbf{r})}\frac{\partial K_x(\mathbf{r})}{\partial x},\quad J_y(\mathbf{r}) = -\frac{1}{\Sigma(\mathbf{r})}\frac{\partial K_y(\mathbf{r})}{\partial y},\quad J_z(\mathbf{r}) = -\frac{1}{\Sigma(\mathbf{r})}\frac{\partial K_z(\mathbf{r})}{\partial z} \tag{13}$$

so that Eq. (11) may be written in standard form as:

$$\nabla\cdot J(\mathbf{r})+\Sigma_{abs}(\mathbf{r})\phi_0(\mathbf{r}) = Q(\mathbf{r})$$

Let us now compare Eqs. (10) and (13), where for consistency:

$$\begin{aligned}K_x(\mathbf{r}) &= \int d\mathbf{\Omega}\Omega_x^2\phi(\mathbf{r},\mathbf{\Omega}) = \sum_{a=1}^{N}w_a\mu_a^2\varphi_a(\mathbf{r})\\ K_y(\mathbf{r}) &= \int d\mathbf{\Omega}\Omega_y^2\phi(\mathbf{r},\mathbf{\Omega}) = \sum_{a=1}^{N}w_a\mu_a^2\varphi_a(\mathbf{r})\\ K_z(\mathbf{r}) &= \int d\mathbf{\Omega}\Omega_z^2\phi(\mathbf{r},\mathbf{\Omega}) = \sum_{a=1}^{N}w_a\mu_a^2\varphi_a(\mathbf{r})\end{aligned} \tag{14}$$

which implies that $K_x = K_y = K_z$. Also if we add the equations above we get:

$$\int d\mathbf{\Omega}(\Omega_x^2+\Omega_y^2+\Omega_z^2)\phi(\mathbf{r},\mathbf{\Omega}) = \int d\mathbf{\Omega}\phi(\mathbf{r},\mathbf{\Omega}) = \phi_0(\mathbf{r}) = 3\sum_{\alpha a=1}^{N}w_a\mu_a^2\varphi_a(\mathbf{r}) \tag{15}$$

which from Eq. (4) is clearly incorrect. Where is the problem? It seems to be connected with the A_N assumption of using the same μ_a for each direction. While K_z is acceptable with $\Omega_z=\mu$ and hence μ_a, the x and y cases would need $\sqrt{1-\mu_a^2}\cos\psi_b$ and $\sqrt{1-\mu_a^2}\sin\psi_b$ so that the sum adds up to unity. We explore this phenomenon in the following section, but it is important to note once again that the authors of the A_N method did not attach any physical meaning to μ_a. It is only in this work that we try to do this by using μ_a as an intuitional guide.

3. Modifications to A_N method

As a possible method of approach, we could modify the procedure described above by associating a different value of μ_a with each direction as hinted at in the previous paragraph. To do that we re-write the A_N Eq. (3) as:

$$\mu_{abx}^2\frac{\partial}{\partial x}\frac{1}{\Sigma(\mathbf{r})}\frac{\partial\varphi_{ab}(\mathbf{r})}{\partial x}+\mu_{aby}^2\frac{\partial}{\partial y}\frac{1}{\Sigma(\mathbf{r})}\frac{\partial\varphi_{ab}(\mathbf{r})}{\partial y}+\mu_{abz}^2\frac{\partial}{\partial z}\frac{1}{\Sigma(\mathbf{r})}\frac{\partial\varphi_{ab}(\mathbf{r})}{\partial z} - \Sigma(\mathbf{r})\varphi_{ab}(\mathbf{r})+\Sigma_s(\mathbf{r})\sum_{\bar a=1}^{N}\sum_{\bar b=1}^{M}u_{\bar a}v_{\bar b}\varphi_{\bar a\bar b}(\mathbf{r})+S(\mathbf{r})=0 \tag{16}$$

with:

$$\mu_{abx} = \sqrt{1-\mu_a^2}\cos\psi_b,\quad \mu_{aby} = \sqrt{1-\mu_a^2}\sin\psi_b,\quad \mu_{abz}=\mu_a, \tag{17}$$

and $v_b = 1/M$ and $\psi_b = \pi b/M$. The quantities u_a and μ_a are obtained from the Gauss–Legendre quadrature points. We now define the directional currents:

$$J_{abx}(\mathbf{r}) = -\frac{\mu_{abx}^2}{\Sigma(\mathbf{r})}\frac{\partial}{\partial x}\varphi_{ab}(\mathbf{r}),\quad J_{aby}(\mathbf{r}) = -\frac{\mu_{aby}^2}{\Sigma(\mathbf{r})}\frac{\partial}{\partial y}\varphi_{ab}(\mathbf{r}),\quad J_{abz}(\mathbf{r}) = -\frac{\mu_{abz}^2}{\Sigma(\mathbf{r})}\frac{\partial}{\partial z}\varphi_{ab}(\mathbf{r}) \tag{18}$$

with:

$$\mathbf{J}_{ab} = \mathbf{i}J_{abx}+\mathbf{j}J_{aby}+\mathbf{k}J_{abz} \tag{19}$$

Multiplying Eq. (18) by suitable weight functions u_a and v_b and summing over a and b, we find:

$$\begin{aligned}\Sigma(\mathbf{r})J_x(\mathbf{r}) &= -\tfrac{\partial}{\partial x}\sum_{a=1}^{N}\sum_{b=1}^{M}u_a v_b\mu_{abx}^2\varphi_{ab}(\mathbf{r})\\ \Sigma(\mathbf{r})J_y(\mathbf{r}) &= -\tfrac{\partial}{\partial y}\sum_{a=1}^{N}\sum_{b=1}^{M}u_a v_b\mu_{aby}^2\varphi_{ab}(\mathbf{r})\\ \Sigma(\mathbf{r})J_z(\mathbf{r}) &= -\tfrac{\partial}{\partial z}\sum_{a=1}^{N}\sum_{b=1}^{M}u_a v_b\mu_{abz}^2\varphi_{ab}(\mathbf{r})\end{aligned} \tag{20}$$