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Derivation of the Green's function of the linear transport equation from the discrete-ordinate solution

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ABSTRACT

A new, simple way to calculate the Green's function of the linear transport equation in a homogeneous, infinite medium is presented. The solution resorts to the corresponding Green's function for the discrete-ordinate method, which is readily obtainable, by using a suitable procedure to calculate the limit when the number of discrete directions tends to infinity.

1. Introduction

The calculation of the Green's function of the linear transport equation for a homogeneous, infinite medium is a classic problem, whose exact solution can be obtained using the Fourier transform method (Bell and Glasstone, 1970) or by the Case technique (Case and Zwefel, 1967) (the equivalence between the two mathematical methods has been analyzed in detail in Ganapol, 2000). In the present paper, the solution is obtained starting from the Green's function corresponding discrete-ordinate solution (Ronen, 1986; Coppa and Ravetto, 1980). More specifically, the transport equation is discretized with respect to the angular variable, so obtaining a set of ordinary differential equation, which can be solved analytically. The original contribution of the paper is the exact calculation of the transient flux, which is evaluated starting from the discrete-ordinate solution in the limit when the number of directions tends to infinity.

The paper is organized as follows. In Section 2, the discrete-ordinate solution is provided in a compact way; in Section 3, the problems arising when the number of directions tends to infinity are presented, together with the calculation of the asymptotic flux. Finally, the evaluation of the transient flux is described in Section 4. Some concluding remarks are drawn in the final section.

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2. Green's function for the discrete-ordinate approximation of the linear transport equation

In the work, an homogeneous, infinite medium is considered. The stationary transport equation in plane geometry for a unitary source located at $x = 0$ can be written (in dimensionless units) as

$$\mu \frac{\partial \varphi}{\partial x} + \varphi = \frac{\gamma \Phi + \delta(x)}{2}, \tag{1}$$

where $\varphi(x, \mu)$ is the angular flux and $\Phi(x)$ the total flux, defined as:

$$\Phi(x) = \int_{-1}^{+1} \varphi(x, \mu) \mathrm{d}\mu. \tag{2}$$

The constant γ, which represents the ratio between the scattering and the total cross sections, is assumed belonging to the interval (0,1). According to the discrete ordinate method, the integral appearing in Eq. (2) is approximated by using a suitable quadrature formula, as

$$\Phi(x) = \sum_{i=1}^{N} \varphi(x, \mu_i) w_i, \tag{3}$$

where μ_i and w_i are the abscissae and the weights of an N-point formula for the numerical integration in the interval $[-1, 1]$. Therefore, the calculation of $\Phi(x)$ requires to evaluate the set of angular fluxes $\varphi_i(x) = \varphi(x, \mu_i)$, which are solutions of the transport Eq. (1) for $\mu = \mu_i$, and the accuracy of the method is expected to increase with the number N of directions for which the angular flux is evaluated.

Usually the set $\{\mu_i\}$ is chosen in a symmetric way, i.e., both μ_i and $-\mu_i$ belong to the set of abscissae. In other terms, the set $\{\mu_i\}$ can be written as

$$\{\mu_i\} = \{-\mu_n, -\mu_{n-1}, \dots, -\mu_1, \mu_1, \dots, \mu_{n-1}, \mu_n\} \tag{4}$$

with

$$0 < \mu_1 < \mu_2 < \dots < \mu_n \le 1. \tag{5}$$

Moreover, μ_i and $-\mu_i$ are associated a same weight, w_i. With this choice, it turns out useful introducing an index p running from $-n$ to -1 and from 1 to n, assuming $\mu_{-p} = -\mu_p$ and $w_{-p} = w_p$, $p = 1, 2, \dots, n$. Moreover, the weights w_p are non negative and $w_1 + w_2 + \dots + w_n = 1$. The angular fluxes φ_p and φ_{-p} satisfy the following differential equations

$$\mu_p \frac{d\varphi_p}{dx} + \varphi_p = \frac{\gamma\Phi + \delta(x)}{2}, \quad -\mu_p \frac{d\varphi_{-p}}{dx} + \varphi_{-p} = \frac{\gamma\Phi + \delta(x)}{2} \qquad p = 1, 2, \dots, n. \tag{6}$$

By summing and subtracting Eq. (6), one obtains

$$\mu_p \frac{d\left(\varphi_p - \varphi_{-p}\right)}{dx} + F_p = \gamma\Phi + \delta(x) \tag{7}$$

and

$$\mu_p \frac{dF_p}{dx} + \left(\varphi_p - \varphi_{-p}\right) = 0, \tag{8}$$

having defined

$$F_p(x) = \varphi_p(x) + \varphi_{-p}(x), \qquad p = 1, 2, \dots, n. \tag{9}$$

Finally, eliminating $\varphi_p - \varphi_{-p}$ from Eqs. (7) and (8), one obtains

$$\mu_p^2 \frac{d^2F_p}{dx^2} - F_p + \gamma\Phi + \delta(x) = 0, \qquad p = 1, 2, \dots, n. \tag{10}$$

The total flux can be expressed in terms of the set $\{F_p\}$, as

$$\Phi = \sum_{p=1}^{n} w_p F_p. \tag{11}$$

In this way, the calculation of the flux is reconducted to the solution of the following system of n second-order differential equations:

$$\mu_p \frac{d^2F_p}{dx^2} - F_p + \gamma \sum_{q=1}^{n} F_q w_q + \delta(x) = 0. \tag{12}$$

The change of variables

$$F_p = \frac{1}{\mu_p \sqrt{w_p}} f_p \tag{13}$$

allows one to write the system in a more suitable form, as

$$\frac{d^2f_p}{dx^2} - \frac{f_p}{\mu_p^2} + \gamma \sum_{q=1}^{n} \frac{\sqrt{w_p w_q}}{\mu_p \mu_q} f_q + \frac{\sqrt{w_p}}{\mu_p} \delta(x) = 0, \tag{14}$$

or, in vector form, as

$$\frac{d^2\mathbf{f}}{dx^2} - \Theta\mathbf{f} + \boldsymbol{\sigma}\delta(x) = 0, \tag{15}$$

where

$$\Theta_{pq} = \frac{\delta_{pq}}{\mu_p^2} - \gamma \frac{\sqrt{w_p w_q}}{\mu_p \mu_q}, \quad \sigma_p = \frac{\sqrt{w_p}}{\mu_p}. \tag{16}$$

The analytic solution of Eq. (15) can be readily obtained after evaluating eigenvalues and eigenvectors of Θ. If λ_α is an eigenvalue of Θ, the components V_p^α of the corresponding eigenvector can be written as

$$V_p^\alpha = \frac{\gamma \mu_p \sqrt{w_p}}{1 - \lambda_\alpha \mu_p^2} A_\alpha, \quad p = 1, 2, \dots, n. \tag{17}$$

where the quantity

$$A_\alpha = \sum_{q=1}^{n} \frac{\sqrt{w_q} V_q^\alpha}{\mu_q} \tag{18}$$

represents a multiplicative constant appearing in the definition of each eigenvector. The constant is arbitrary, but multiplying Eq. (17) by $\sqrt{w_p}/\mu_p$ and summing over p the result must be A_α again: this is possible if and only if the condition

$$\sum_{p=1}^{n} \frac{w_p}{1 - \lambda_\alpha \mu_p^2} = \frac{1}{\gamma} \tag{19}$$

is satisfied. As Θ is hermitian, its eigenvalues are all real numbers (and the corresponding eigenvectors are mutually orthogonal). It can be readily verified, e.g. in a graphic way by considering in the plane (λ, ν) the intersection of the curve

$$\nu = \sum_{p=1}^{n} \frac{w_p}{1 - \lambda \mu_p^2} \tag{20}$$

with the straight line $\nu = \frac{1}{\gamma}$, that n distinct, positive eigenvalues $\lambda_\alpha = k_\alpha^2$ exist for $\gamma < 1$. Moreover, each of the intervals I_α, defined as

$$I_1 = \left(\frac{1}{\mu_2}, \frac{1}{\mu_1}\right), \; I_2 = \left(\frac{1}{\mu_3}, \frac{1}{\mu_2}\right), \dots, \; I_n = \left(0, \frac{1}{\mu_n}\right) \tag{21}$$

contains one of the solutions k_α, i.e., $k_\alpha \in I_\alpha$. Consequently, the eigenvectors $\{\boldsymbol{V}^\alpha\}$ represent an orthogonal base for $\mathbb{R}^n$. In the following, such vectors are supposed to be normalized, i.e.: $\boldsymbol{V}^\alpha \cdot \boldsymbol{V}^\beta = \delta_{\alpha\beta}$. To do that, the constant A_α of Eq. (17) relative to the eigenvalue k_α^2 must be set as

$$A_\alpha = \left\{ \gamma^2 \sum_{p=1}^{n} \frac{\mu_p^2 w_p}{\left(1 - k_\alpha^2 \mu_p^2\right)^2} \right\}^{-\frac{1}{2}}. \tag{22}$$

After calculating eigenvalues and eigenvectors of Θ, the solution of Eq. (15) can be written as

$$\mathbf{f}(x) = \sum_{\alpha=1}^{n} \left[\mathbf{f}(x) \cdot \mathbf{V}^\alpha\right] \mathbf{V}^\alpha, \tag{23}$$

where the projection of $\boldsymbol{f}$ on $\boldsymbol{V}^\alpha$ satisfies the equation:

$$\frac{d^2(\mathbf{f} \cdot \mathbf{V}_\alpha)}{dx^2} - k_\alpha^2 (\mathbf{f} \cdot \mathbf{V}^\alpha) + (\boldsymbol{\sigma} \cdot \mathbf{V}^\alpha)\delta(x) = 0 \tag{24}$$

with the boundary conditions

$$\lim_{x \to \pm\infty} \mathbf{f}(x) \cdot \mathbf{V}^\alpha = 0. \tag{25}$$

From the solution of Eqs. (24), (25),

$$\mathbf{f}(x) \cdot \mathbf{V}^\alpha = \frac{(\boldsymbol{\sigma} \cdot \mathbf{V}^\alpha)}{2k_\alpha} \exp(-k_\alpha |x|), \tag{26}$$

one can finally calculate the total flux, so obtaining

$$\Phi(x) = \sum_{p=1}^{n} \frac{\sqrt{w_p}}{\mu_p} f_p(x) = \boldsymbol{\sigma} \cdot \mathbf{f}(x) = \sum_{\alpha=1}^{n} \frac{(\boldsymbol{\sigma} \cdot \mathbf{V}^\alpha)^2}{2k_\alpha} \exp(-k_\alpha |x|). \tag{27}$$

Noticing that

$$\boldsymbol{\sigma} \cdot \mathbf{V}^\alpha = A_\alpha \sum_{p=1}^{n} \frac{\gamma w_p}{1 - \lambda_\alpha \mu_p^2} = A_\alpha, \tag{28}$$

and defining the constants C_α as