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On the second moment of a stochastic radiation field

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ABSTRACT

A space dependent model for the second moment of a stochastic field has been presented. The derivation procedure is simple and avoids non-physical assumptions for completeness. Moreover, a program has been developed which obtains the solutions to this equation in the three dimensional space. Two benchmarks, namely the fissile cube and the subcritical slab, show the accuracy of the results obtained using this method.

1. Introduction

The basic interactions of fundamental particles, e.g. photons and neutrons, with a media is stochastic in nature. This stochastic behavior is described by the quantum physical laws (Zettili, 2009) and is manifest in the interaction cross sections (Berger et al., 1999; Chadwick et al., 2006), i.e. probability of interaction per unit length. The cumulative result of these stochastic interactions is a stochastic field. While a genuine study of nature's behavior requires the inspection of the evolution of the probability measure of this stochastic field throughout space and time, this aim remains a computational challenge, even for the lowest density fields, for the near future. As a result, this stochastic field has been studied with various degrees of accuracy and simplicity. Some of these works are physically oriented (Huang et al., 2008; Lewins, 1978; Osborn and Yip, 1966), while others seek mathematical approaches to gain further simplicity (Allen, 2012; Pázsit and Pál, 2008; Pluta, 1962).

A well-known method in the investigation of stochastic fields is the study of its moments (Bell, 1969). Substantive work has been done on the study of the first moment of this stochastic field, which is widely known as the Boltzmann equation (Altaç and Tekkalmaz, 2002; Ayyoubzadeh and Vosoughi, 2011; Chandrasekhar, 2013; Prahl, 1988). While these studies gain insight into the original field by investigating its average behavior, they are far from explaining all the measurable quantities of the field. Hence, it might seem natural to harvest further information of the field by studying it's second (and higher order) moments, i.e. the variance and covariance. As a result, a measurable amount of work has been directed to the study of these quantities (Degweker, 1994; Edelmann et al., 1975; Hayes and Allen, 2005; Muñoz-Cobo et al., 2000; Muñoz-Cobo and Rugama, 2003; Sharp and Allen, 2000). The well-known Feynman-alpha and Rossi-alpha methods are notable results of such studies (Ceder and Pázsit, 2003; Otsuka and Iijima, 1965; Pál and Pázsit, 2012; Pázsit and Pál, 2008; Pázsit and Yamane, 1999).

A considerable share of the investigation of the second order moments of the stochastic field has been focused on the one point approximation in which one neglects spatial, i.e. transport, effects. Furthermore, the inclusion of these effects has been mostly accompanied by the backward approach which in our view overshadows the underlying physics. While forward approach derivations seem to be available (Degweker, 1994), the complexity of the mathematical apparatus used in these studies discourages their routine use by researchers. Moreover, numerical results for these equations seem scarce.

The current paper is organized in eight sections. In the next section, the stochastic process underlying a transport phenomenon is quantified using a probability measure. The third section investigates the evolution of this measure throughout space and time. This derivation has been based on a forward approach and differs from the probability measure evolution equation in (Degweker, 1994) in one term. In section four, the equations governing the first and second order moments have been derived using the result of the previous section. The use of the moment generating functional

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method (Degweker, 1994) has been avoided to achieve simplicity. Furthermore we have refrained from use of questionable assumptions, i.e. the extended boundary method, which aid clearness. The obtained equation differs from (Degweker, 1994) in the source terms. In section five, a six-dimensional spatial element has been introduced and the second order evolution equation has been discretized in this element. In the sixth section, the outline of a program developed to solve this equations have been pointed out. The seventh section includes two numerical benchmarks which show the compatibility of the results with results obtained from direct Monte Carlo simulations. Finally, the results are discussed and improvements have been suggested.

2. The stochastic field

We shall assume each particle in the radiation field to be fully describable by a point in the phase space X which includes the position, speed and direction of flight of that particle. One may readily notice that the phase space points x are embedded in the field $\mathbb{R}^3 \times \mathbb{R}^+ \times \mathrm{S}$ where $\mathbb{R}$ denotes the real numbers space and $\mathrm{S} \subset \mathbb{R}^3$ denotes the unit sphere. Note that by this definition, X spans the finite volume of the region of interest along with all the speeds and directions of flight that are possible for the particle in this region. For the description of the stochastic field, we need to define a suitable probability measure. In this paper, we shall use the measure introduced in (Degweker, 1994), specifically the probability density of having n particles in the phase space, where the i'th particle position in the phase space is denoted by x_i. Note that the probability of having these particles at infinitesimal volumes $\delta\, x_i$ containing each point x_i at some time t is obtained from $p_n(x_1, \ldots ,x_n;t)\delta x_1 \ldots \delta x_n$ where p_n is the defined measure, i.e. a probability density function. Here, we shall not be concerned about the technicalities of the probability space under consideration and assume it to be well-defined. Having defined this measure, the most prominent properties of the stochastic field under study, namely the mean and covariance could be obtained. To do so, one may note that the total probability of having one particle at an infinitesimal volume δx_1 containing the point x_i is the sum of having just one particle in the total space, where this particle is in the infinitesimal volume under study plus the probability of having more than one particles where one of them is placed in the desired volume. Showing the probability of having one particle in this volume by $f_1(x_1;t)\delta x_1$, this statement could be written as

$$f_1(x_1;t)\delta x_1 = p_1(x_1;t)\delta x_1 + \sum_{n=2}^{\infty} \frac{\delta x_1}{(n-1)!} \int_{x_2\in X} \cdots \int_{x_n\in X} p_n(x_1,\ldots,x_n;t)dx_n \ldots dx_2. \quad (1)$$

Now one may note that the mean number of particles in this infinitesimal volume could be obtained by

$$E(N_{\delta x_1}) = \sum_{n_1=0}^{\infty} n_1 \mathrm{Pr}_{n_1}(x_1;t)(\delta x_1)^{n_1}, \quad (2)$$

where in Eq. (2), $E()$ is to be understood as the expected value operator, i.e. an operator which gives the mean value of the operand (Papoulis and Pillai, 2002). Also, $N_{\delta x_1}$ is a random variable denoting the number of particles in the volume δx_1 and $\mathrm{Pr}_{n_1}(x_1;t)$ is the probability density of having n_1 particles at x_1. Note that while $\mathrm{Pr}_1(x_1;t)$ is equal to $f_1(x_1;t)$, we have no information about $\mathrm{Pr}_{n_1}(x_1;t)$ for $n_1 > 1$. However, using the assumption that δx_1 is an infinitesimal volume one could use Eq. (2) to obtain the mean density of particles at point x_1 as

$$\bar{n}(x_1) = \lim_{\delta x_1 \to 0} E\left(\frac{N_{\delta x_1}}{\delta x_1}\right) = \mathrm{Pr}_1(x_1;t) = f_1(x_1;t). \quad (3)$$

One may use a similar argument to obtain the probability of having one particle inside the infinitesimal volume δx_1 and another particle inside the infinitesimal volume δx_2 from the measure p_n as

$$f_2(x_1,x_2;t)\delta x_1 \delta x_2 = p_2(x_1,x_2;t)\delta x_1 \delta x_2 + \sum_{n=3}^{\infty} \frac{\delta x_1 \delta x_2}{(n-2)!} \int_{x_3\in X} \cdots \int_{x_n\in X} p_n(x_1,x_2,\ldots,x_n;t)dx_n \ldots dx_3. \quad (4)$$

Note that the expected value of the number of particles inside $\delta\, x_1$ and δx_2 could be obtained from

$$E(N_{\delta x_1} N_{\delta x_2}) = f_2(x_1,x_2;t)\delta x_1 \delta x_2 + O(\delta x_1^2)\delta x_2 + O(\delta x_2^2)\delta x_1, \quad (5)$$

where in Eq. (5), $O()$ denotes the order of magnitude of a term. Using the assumption that δx_1 and δx_2 are infinitesimal volumes, Eq. (5) reduces to

$$\lim_{\substack{\delta x_1 \to 0 \\ \delta x_2 \to 0}} \frac{E(N_{\delta x_1} N_{\delta x_2})}{\delta x_1 \delta x_2} = f_2(x_1,x_2;t). \quad (6)$$

Here we point out that the covariance of the particle density between two points in the phase space, which describes the fluctuations of the stochastic field to some degree, is obtained from

$$\begin{aligned}\mathrm{cov}(x_1,x_2;t) &= E\left(\lim_{\substack{\delta x_1 \to 0 \\ \delta x_2 \to 0}} \frac{N_{\delta x_1} N_{\delta x_2}}{\delta x_1 \delta x_2}\right) - E\left(\lim_{\delta x_1 \to 0} \frac{N_{\delta x_1}}{\delta x_1}\right) E\left(\lim_{\delta x_2 \to 0} \frac{N_{\delta x_2}}{\delta x_2}\right) \\ &= f_2(x_1,x_2;t) - f_1(x_1;t) f_1(x_2;t). \end{aligned} \quad (7)$$

In the next section we shall describe a method for obtaining p_n in a media with known stochastic properties.

3. The probability measure evolution

In this section the probability measure evolution is obtained by improving the method described in (Degweker, 1994). To obtain such an equation, we shall assume that the particle interacts with the media at discrete locations and that the type and cross section of events, i.e. probability of occurrence per unit length, is as summarized in Table 1.

Note that while $\Sigma_{\mathrm{f},m}(x_i \to x_{f_1}, \ldots, x_{f_m})$ is mainly used to describe a fission which yields m-neutron, it could contain other processes such as the multiple-photon emission process (Albota et al., 1998; Xu and Webb, 1996) in the optical realm. Also, it has been assumed that an external source is placed at x_i which emits de-correlated particles one at a time with a constant probability $S(x_i)\delta t$. Now, by assuming $p_n(x_1, \ldots ,x_n;t)$ to be known, at an infinitesimal later time $t + \delta t$ one may write

Table 1
Probability of various interactions for the particle with the surrounding media.

Type of interaction	Probability of occurrence per unit length
Particle scattering from x_i to x_f	$\Sigma_{\mathrm{s}}(x_i \to x_f)$
Particle capture at x_i	$\Sigma_{\mathrm{c}}(x_i)$
Particle absorbed at x_i to yield m new particles at $\{x_{f_1}, \ldots, x_{f_m}\}$	$\Sigma_{\mathrm{f},m}(x_i \to x_{f_1}, \ldots, x_{f_m})$
Particle escape from boundary	$\begin{cases} 1, & \text{If particle is on boundary and flying outwards} \\ 0, & \text{Otherwise} \end{cases}$