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Analytical benchmark for non-equilibrium radiation diffusion in finite size systems

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ABSTRACT

Non-equilibrium radiation diffusion is an important mechanism of energy transport in inertial confinement fusion, astrophysical plasmas, furnaces and heat exchangers. In this paper, an analytical solution to the non-equilibrium Marshak diffusion problem in a planar slab and spherical shell of finite thickness is presented. Using Laplace transform method, the radiation and material energy densities are obtained as functions of space and time. The variation in integrated energy densities and leakage currents are also studied. In order to linearize the radiation transport and material energy equation, the heat capacity is assumed to be proportional to the cube of the material temperature and the opacity to be independent of temperature. The steady state energy densities show linear variation along the depth of the planar slab, whereas non-linear dependence is observed for the spherical shell. The analytical energy densities show good agreement with those obtained from finite difference method using small mesh width and time step. The benchmark results obtained in this work can be used to validate and verify non-equilibrium radiation diffusion computer codes in both planar and spherical geometry.

1. Introduction

The time dependent non-equilibrium radiation transport equation is non-linearly coupled to the material energy equation (Pomraning, 1973; Mihalas and Mihalas, 1984). Also the material properties have complex dependence on the independent variables. As a result, the time dependent thermal radiation transport problems are commonly solved numerically. Several numerical methods are in use for this purpose, namely the discrete ordinates (Mishra et al., 2006, 2011; Ghosh and Menon, 2010), finite volume (Kim et al., 2010), Monte Carlo (Fleck and Cummings, 1971), hybrid stochastic-deterministic (Densmore, 2006; Connolly et al., 2012), or the approximate methods like the Eddington approximation (Shettle and Weinman, 1970), heat conduction (Goldstein, 2010) or the diffusion approximations (Dai and Woodward, 1998; Knoll et al., 2001; Ober and Shadid, 2005). Benchmark results for test problems are necessary to validate and verify the numerical codes (Ensman, 1994). Analytical solutions producing explicit expressions for the radiation and material energy density, integrated densities, leakage currents, etc. are the most desirable.

In the literature, considerable amount of efforts have been applied for solving the radiation transport problem analytically. Marshak obtained a semi-analytical solution by considering radiation diffusion in a semi-infinite planar slab with radiation incident upon the surface (Marshak, 1958). Assuming that the radiation and material fields are in equilibrium, the problem admits a similarity solution to a second order ordinary differential equation which was solved numerically (Kass and O'Keeffe, 1966). The results were extended for non-equilibrium radiation diffusion by assuming that the specific heat is proportional to the cube of the temperature and the opacity is constant (Pomraning, 1979; Su and Olson, 1996). This assumption linearized the problem providing a detailed analytical solution. As the radiative transfer codes are meant to handle an arbitrary temperature dependence of the material properties, the obtained solutions serve as a useful test problem (Ganapol and Pomraning, 1983; Su and Olson, 1997, 1999). Using the same linearization, 3T radiation diffusion equations were solved for spherical sources in an infinite medium (McClarren and Wohlbier, 2011). All available results on the non-equilibrium radiative transfer problems in planar and spherical geometry consider systems having infinite or semi-infinite extension. Benchmarks involving finite size systems have been limited either to the heat conduction or equilibrium diffusion approximation (Williams, 2005; Olson and Henderson, 2004; Liemert and Kienle, 2012).

In this paper, we solve the time dependent non-equilibrium radiation diffusion problem for finite size systems in both planar and spherical geometry. The coupled system of radiation diffusion and material energy equations have been linearized by assuming the opacity to be independent of temperature and the specific heat to be proportional to the cube of the material temperature. Non-equilibrium diffusion codes can be more easily validated

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and verified against these benchmark results because there is no need to consider a slab or spherical medium of very large size for avoiding boundary effects. Analytical expressions for all the quantities of interest can be obtained for finite slab/shell width and parameter values relevant to practical problems. This work can be extended to multi-dimension using separation of variables and Laplace transform method or the eigenfunction expansion method to obtain analytical series solution in a manner similar to the multilayer heat conduction (Jain et al., 2010).

The remainder of the paper is organized as follows. In Section 2, the analytical solution for the finite planar slab and spherical shell is derived followed by Section 3 on numerical finite difference method. In Section 4, the results for the radiation and material energy densities, leakage currents, integrated quantities, etc. are plotted and physically explained. Finally, conclusions are given in Section 5.

2. Analytical solution

A radiation field in space is described by the distribution of the intensity of radiation w.r.t., frequency ν, to space $\vec{r}$, to the direction of radiant energy transfer $\vec{\Omega}$ and is expressed in terms of the intensity of radiation $I(\nu,\vec{r},\vec{\Omega},t)$ (Zeldovich and Raizer, 1966). The time dependent, multi-frequency, non-equilibrium, classical radiation transport equation (RTE) is given by (Pomraning, 1973)

$$\frac{1}{c}\frac{\partial I(\nu,\vec{r},\vec{\Omega},t)}{\partial t}+\vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{r},\vec{\Omega},t)=S(\nu,\vec{r},t)-\kappa(\nu,\vec{r},t)I(\nu,\vec{r},\vec{\Omega},t)+\int_0^\infty d\nu'\int_{4\pi}d\vec{\Omega}'\left[\frac{\nu}{\nu'}\sigma_s(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega})I(\nu',\vec{\Omega}')-\sigma_s(\nu\to\nu',\vec{\Omega}\cdot\vec{\Omega}')I(\nu,\vec{\Omega})\right], \quad (1)$$

where κ is the opacity (absorption cross-section), σ_s is the scattering cross-section, c is the speed of light and $S(\nu,\vec{r},t)$ denotes the rate of energy emission due to spontaneous processes.

Eq. (1) is an integro-differential equation and because of its partly differential character, requires both spatial and temporal boundary conditions. Also, it is very complicated as the dependent variable, the specific intensity depends upon seven independent variables. Hence, in general, to obtain analytical solutions, it is necessary to make some approximations. Firstly, the concept of local thermodynamic equilibrium (LTE) assumption is introduced. It is assumed that the properties of the matter are dominated by atomic collisions which establish thermodynamic equilibrium locally at position $\vec{r}$ and time t, and the radiation field, even if it deviates substantially from the equilibrium Planck distribution, does not affect this equilibrium. The state of the material is then described by two parameters, namely the temperature and density. The emission term is then given by

$$S(\nu,\vec{r},t)=\kappa(\nu,\vec{r},t)B(\nu,\vec{r},t),$$

with Planck's function $B(\nu,\vec{r},t)=\frac{2h\nu^3}{c^2}\left(\exp\left(\frac{h\nu}{k_BT(\vec{r},t)}\right)-1\right)^{-1}$. Here, k_B is Boltzmann's constant, h is Planck's constant and $T(\vec{r},t)$ is the local material temperature. The next approximation is the Grey approximation in which the opacities are assumed to be frequency independent so that the RTE may be integrated over all frequencies. If the specific intensity of radiation is almost isotropic, the diffusion approximation can be used. It is applicable for optically thick bodies where the gradients of radiation energy density are small. The basic assumption underlying the diffusion approximation for radiative transfer is that the angular dependence of the specific intensity can be represented by the first two terms in a spherical harmonic expansion. Neglecting hydrodynamic motion and restricting the medium to be an ideal fluid (no viscous or heat conduction effect), the one group radiative transfer equation in the diffusion approximation and the material energy (ME) balance equation on neglecting the scattering terms are (Pomraning, 1973)

$$\frac{1}{c}\frac{\partial E(\vec{r},t)}{\partial t}-\vec{\nabla}\cdot D\vec{\nabla}E(\vec{r},t)=\kappa(\vec{r},t)[aT^4(\vec{r},t)-E(\vec{r},t)], \quad (2)$$

$$C_\nu(T)\frac{\partial T(\vec{r},t)}{\partial t}=c\kappa(\vec{r},t)[E(\vec{r},t)-aT^4(\vec{r},t)], \quad (3)$$

where $E(\vec{r},t)$ is the radiation energy density, $D=1/(3\kappa(\vec{r},t))$ is the diffusion coefficient, a is the radiation constant, and $C_\nu(T)$ is the specific heat of the material.

2.1. Planar slab

We consider a planar slab of finite thickness which is purely absorbing and homogeneous occupying $0\leqslant z\leqslant l$. The medium is at zero temperature initially. At time $t=0$, a constant diffuse radiative flux (F_{inc}) is incident on the surface at $z=0$ as shown in Fig. 1. The one dimensional planar radiation diffusion equation along with the material energy balance equation are (Pomraning, 1973)

$$\frac{\partial E(z,t)}{\partial t}-\frac{\partial}{\partial z}\left[\frac{c}{3\kappa}\frac{\partial E(z,t)}{\partial z}\right]=c\kappa[aT^4(z,t)-E(z,t)], \quad (4)$$

$$C_\nu(T)\frac{\partial T(z,t)}{\partial t}=c\kappa[E(z,t)-aT^4(z,t)]. \quad (5)$$

To remove the nonlinearity in the radiation diffusion (Eq. (4)) and material energy equation (Eq. (5)), opacity κ is assumed to be independent of temperature and specific heat C_ν is assumed to be proportional to the cube of the temperature i.e., $C_\nu=\alpha T^3$. The Marshak boundary condition on the surface at $z=0$ is given by

$$E(0,t)-\left(\frac{2}{3\kappa}\right)\frac{\partial E(0,t)}{\partial z}=\frac{4}{c}F_{inc}. \quad (6)$$

And that at $z=l$ is

$$E(l,t)+\left(\frac{2}{3\kappa}\right)\frac{\partial E(l,t)}{\partial z}=0. \quad (7)$$

The initial conditions on these two equations are

$$E(z,0)=T(z,0)=0. \quad (8)$$

The RTE and the ME are recast into the dimensionless form by introducing the dimensionless independent variables given by

$$x\equiv\sqrt{3}\kappa z,\ \tau\equiv\left(\frac{4ac\kappa}{\alpha}\right)t, \quad (9)$$

and new dependent variables given by

$$u(x,\tau)\equiv\left(\frac{c}{4}\right)\left[\frac{E(z,t)}{F_{inc}}\right],\ v(x,\tau)\equiv\left(\frac{c}{4}\right)\left[\frac{aT^4(z,t)}{F_{inc}}\right]. \quad (10)$$

With these new variables, the RTE and ME take the dimensionless form

$$\varepsilon\frac{\partial u(x,\tau)}{\partial\tau}=\frac{\partial^2u(x,\tau)}{\partial x^2}+v(x,\tau)-u(x,\tau), \quad (11)$$

$$\frac{\partial v(x,\tau)}{\partial\tau}=u(x,\tau)-v(x,\tau), \quad (12)$$

with the initial conditions

$$u(x,0)=0, \quad (13)$$

$$v(x,0)=0. \quad (14)$$

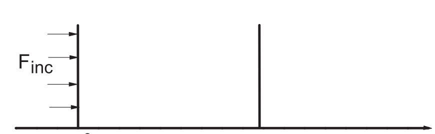

Fig. 1. Diffuse radiation flux incident on the left surface of a slab of thickness $z=l$.