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A highly accurate algorithm for the solution of the point kinetics equations

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ABSTRACT

Attempts to resolve the point kinetics equations (PKEs) describing nuclear reactor transients have been the subject of numerous articles and texts over the past 50 years. Some very innovative methods, such as the RTS (Reactor Transient Simulation) and CAC (Continuous Analytical Continuation) methods of G.R. Keepin and J. Vigil respectively, have been shown to be exceptionally useful. Recently however, several authors have developed methods they consider accurate without a clear basis for their assertion. In response, this presentation will establish a definitive set of benchmarks to enable those developing PKE methods to truthfully assess the degree of accuracy of their methods. Then, with these benchmarks, two recently published methods, found in this journal will be shown to be less accurate than claimed and a legacy method from 1984 will be confirmed.

1. My concern

The solution to the point kinetics equations, central to nuclear reactor transient analysis, is a challenge that has yet to be satisfactory resolved. There have been several notable historic attempts including those of Nobrega (1971), Keepin (1965), Vigil (1967), Izumi and Noda (1971) and others who, in effect, have provided only modest improvements (if any) over these original methods. In several recent publications however, we find authors presenting numerical schemes with little or no change from those presented earlier. In addition, they have made claims of accuracy that are not based on true benchmarks, but on ad hoc comparisons to other methods. We find numerical values, published to four, five or more places of inferred accuracy, when indeed this is not the case. These articles go under the titles of an *efficient technique* (Nahla, 2011; Aboanber and Hamada, 2002, 2003), *analytical solution* (Nahla, 2010; Petersen et al., 2011), *analytical exponential method* (Nahla, 2008) and *analytical inversion method* (Aboanber and Nahla, 2002). In reality, none of the methods listed are either analytical or truly efficient or, more importantly, of benchmark accuracy defined here to be a consistent 5–9 places.

As an example, we focus on an article entitled "An efficient technique for the point kinetics equations with Newtonian temperature feedback effects" (Nahla, 2011). In this work, we find an ad hoc separation of the non-linear reactivity identical to that found in (Kinard and Allen, 2004). There follows a finite difference development that uses an implicit backward Euler (BE) finite difference (FD) scheme to give an explicit extrapolated value of the unknown (non-linear) reactivity at the next time interval. We then return to the analytical form of the original equation with the (known) advanced reactivity inserted and to be solved analytically, over a time step, as a set of 7 ODEs now with constant reactivity. In the process, there is a rather unorthodox solution to the inhour equation that seems overly complicated and entirely unnecessary, given that one is simply solving for the zeros of a polynomial. The author then compares his results with standard step input and ramp results quoting 5 digits that are in excellent agreement with the literature. Of particular note, the method is tested only for power transients of less than 32 times nominal and not for severe transients. Finally, there is no self-consistency check reported to ensure internal accuracy. Thus, the method has not been thoroughly verified leaving open the question of its overall performance, especially for severe transients. In the final section of the aforementioned article, there are several tables of neutron densities for the case with temperature feedback. The entries are compared to several other methods, where no two methods agree to more than two places while the author quotes 7 digits (6 places). Clearly, any conclusion regarding the merit of the "efficient technique" has no basis without further evidence. In spite of this, the author concludes.

> *"These comparisons substantiated the accuracy, faster [speed] and the efficiency of the efficient techniques."*

The investigation to follow demonstrates that benchmark accuracy of 9-places or more can come from the same basic BEFD scheme found in (Nahla, 2011). Not only, will the simple BE approximation be used in a most efficient way, but also there will be no need to solve the inhour equation, greatly simplifying the

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overall numerical evaluation. In addition, definitive benchmarks will be established for use as standards to enable truthful statements concerning the accuracy of new methods. It must also be remarked, that, arguably, with establishment of the PKE/BEFD algorithm to be presented, there may be no need for future numerical algorithms to solve the PKEs.

2. The backward Euler finite difference (BEFD) approximation

We begin with the point kinetics equations with feedback for m delayed neutron groups

$$\frac{dN(t)}{dt} = \left[\frac{\rho(t,N)-\beta}{\Lambda}\right]N(t) + \sum_{l=1}^{m}\lambda_l C_l(t) + q(t)$$
$$\frac{dC_l(t)}{dt} = \frac{\beta_l}{\Lambda}N(t) - \lambda_l C_l(t), \quad l = 1,\ldots,m \tag{1a}$$

and generally including Newtonian feedback

$$\rho(t,N) = \rho_0(t) - B\int_0^t dt' N(t') \tag{1b}$$

as our initial consideration.

All the symbols have their usual meanings (N is power or neutron density, C_l is the precursor concentration in group l, ρ is reactivity, q is an external source, β_l is the yield from fission of precursors in delayed group l and β is the total yield, λ_l is the precursor decay constant for group l, Λ is the neutron generation time, B is the absolute value of the temperature coefficient of reactivity and ρ_0 is a prescribed reactivity). Note that Eq. (1b) can also be written as

$$\frac{d\rho(t,N)}{dt} = \frac{d\rho_0(t)}{dt} - BN(t) \tag{2}$$

leading to the more condensed vector form for Eqs. (1) and (2)

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}(t,N(t))\mathbf{y}(t) + \mathbf{q}(t) \tag{3a}$$

with

$$\mathbf{y}(t) \equiv \begin{bmatrix} N(t) \\ C_1(t) \\ \cdots \\ C_l(t) \\ \rho(t) \end{bmatrix}, \quad \mathbf{q}(t) \equiv \begin{bmatrix} q(t) \\ 0 \\ \cdots \\ 0 \\ \dfrac{d\rho_0}{dt} \end{bmatrix} \tag{3b}$$

The Jacobian matrix is

$$\mathbf{A}(t,N(t)) \equiv \begin{bmatrix} (\rho(t,N(t))-\beta)/\Lambda & \lambda_1 & \lambda_2 & \cdots & \lambda_m & 0 \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & \cdots & \cdots & 0 \\ \frac{\beta_2}{\Lambda} & 0 & -\lambda_2 & & & \cdots \\ \cdots & \cdots & & \cdots & & \cdots \\ \frac{\beta_m}{\Lambda} & 0 & \cdots & 0 & -\lambda_m & 0 \\ -B & 0 & \cdots & & \cdots & 0 \end{bmatrix} \tag{3c}$$

and Eq. (3a) is to be solved subject to the following initial conditions:

$$\mathbf{y}(0) \equiv \begin{bmatrix} N(0) \\ \frac{\beta_1}{\lambda_1\Lambda}N(0) \\ \cdots \\ \frac{\beta_m}{\lambda_m\Lambda}N(0) \\ \rho_0(0) \end{bmatrix} \tag{3d}$$

A straightforward derivation of the BEFD approximation begins with the following Taylor series (from above in time) at time t_j ($h = t_{j+1} - t_j$):

$$\mathbf{y}(t_j) = \mathbf{y}(t_{j+1}) - h\left.\frac{d\mathbf{y}(t)}{dt}\right|_{t_{j+1}} + \sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\mathbf{y}^{(k)}(t_{j+1})h^k \tag{4}$$

to also play a primary role below. With substitution of Eqs. (3a) and (4) becomes

$$\mathbf{y}(t_j) = \mathbf{y}(t_{j+1}) - h[\mathbf{A}(t_{j+1},N(t_{j+1}))\mathbf{y}(t_{j+1}) + \mathbf{q}(t_{j+1})] + \sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\mathbf{y}^{(k)}(t_{j+1})h^k$$

The implicit BEFD approximation is then the first three terms of this equation

$$[\mathbf{I} - h\mathbf{A}(t_{j+1},N(t_{j+1}))]\mathbf{y}_{j+1} = \mathbf{y}_j + h\mathbf{q}(t_{j+1})$$

or, on inversion

$$\mathbf{y}_{j+1} = [\mathbf{I} - h\mathbf{A}(t_{j+1},\tilde{N})]^{-1}[\mathbf{y}_j + h\mathbf{q}(t_{j+1})] \tag{5}$$

Note when t_j is in an argument, then the quantity is considered to be exact and to be approximated by the same variable when subscripted.

At this point, we assume that the neutron density $[\tilde{N}]$ in the Jacobian matrix $\mathbf{A}$ is known and represents a first approximation to the fixed-point iteration to be described. As shown by Nobrega (Nobrega, 1971), the sparse $\mathbf{A}$-matrix conveniently lends itself to the analytical inversion

$$[\mathbf{I} - h\mathbf{A}]^{-1} \equiv \frac{1}{\sigma}\begin{bmatrix} 1 \\ \frac{h\beta_1/\Lambda}{1+h\lambda_1} \\ \cdots \\ \frac{h\beta_m/\Lambda}{1+h\lambda_m} \\ -hB \end{bmatrix}\begin{bmatrix} 1 \\ \frac{h\lambda_1}{1+h\lambda_1} \\ \cdots \\ \frac{h\lambda_m}{1+h\lambda_m} \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ 0 & \frac{1}{1+h\lambda_1} & & & \\ \cdots & & \cdots & & \\ & & & \frac{1}{1+h\lambda_m} & 0 \\ 0 & & & 0 & 1 \end{bmatrix}$$

with

$$\sigma \equiv 1 - h\frac{\rho(t_{j+1},\tilde{N})}{\Lambda} + \sum_{l=1}^{m}\frac{h\beta_l/\Lambda}{1+h\lambda_l}.$$

When σ is zero the last expression becomes the inhour equation. We now focus on the numerical algorithm to solve Eq. (5).

3. The PKE/BEFD algorithm and its implementation

The error of the BE approximation can be found by noting that the true solution and approximate solutions are formally related by

$$\mathbf{y}(t_{j+1}) = \mathbf{y}_{j+1} + \boldsymbol{\varepsilon}_{j+1} \tag{6}$$

By expansion of the inverse in small h in the approximate solution to Eq. (5) formally satisfies

$$\mathbf{y}_{j+1} = \mathbf{y}_j + \sum_{k=1}^{\infty}\boldsymbol{\beta}_{kj}h^k \tag{7a}$$

and from Eq. (4)

$$\mathbf{y}(t_{j+1}) = \mathbf{y}(t_j) + \sum_{k=1}^{\infty}\boldsymbol{\gamma}_{kj}h^k. \tag{7b}$$

Subtracting Eqs. (7a) from (7b) and noting Eq. (6) gives

$$\boldsymbol{\varepsilon}_{j+1} = \boldsymbol{\varepsilon}_j + \sum_{k=1}^{\infty}[\boldsymbol{\gamma}_{kj} - \boldsymbol{\beta}_{kj}]h^k \tag{8}$$

Then, on summing over j on $[0, j]$, one finds an explicit form of the error

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