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A problem on functional graded material under fractional order theory of thermoelasticity

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ABSTRACT

The present work is concerned with the solution of a problem on fractional order theory of thermoelasticity for a functional graded material. The governing equations of fractional order generalized thermoelasticity with one relaxation time for functionally graded materials (FGM) (i.e. material with spatially varying material properties) are established. These equations are expressed in Laplace transform domain. The analytical solution in the transform domain is obtained by using the eigenvalue approach. The inversion of Laplace transform is done numerically. Finally, the results obtained are presented graphically to show the effect of the fractional and nonhomogeneity parameters and time on displacement, temperature, and stress.

1. Introduction

There are a number of significant problems in engineering requiring thermal stress analysis. An important class of problems arises in mechanical engineering and includes the analysis of machine components subjected to high temperature environments and large temperature variations such as in a turbine. During recent years, several interesting models have been developed by using fractional calculus to study the physical processes particularly in the area of heat conduction, diffusion, mechanics of solids, electricity, etc. In such cases, one needs to use a generalized thermoelasticity theory based on an anomalous heat conduction model involving time-fractional (non-integer order) derivatives. Abel [1] who applied fractional calculus in the solution of an integral equation gave the first application of fractional derivatives. Caputo [2] gave the definition of fractional derivatives of order $0 < \alpha \leqslant 1$ of continuous function. Caputo and Mainardi [3,4] and Caputo and Mainardi [5] have employed the fractional order derivatives for the description of viscoelastic materials and have established the connection between fractional derivatives and the theory of linear viscoelasticity and found a good agreement with the experimental results. Among the few works devoted to applications of fractional calculus to thermoelasticity, we can refer to the works of Povstenko [6,7], who introduced a fractional heat conduction law, found the associated thermal stresses. Sherief et al. [8], Youssef [9] and Ezzat [10,11] introduced new models of thermoelasticity using a fractional heat conduction equation.

Functionally graded material (FGM) as a new kind of composites were initially designed as thermal barrier materials for aerospace structures, in which the volume fractions of different constituents of composites vary continuously from one side to another Suresh and Mortensen [12]. These novel nonhomogeneous materials have excellent thermo-mechanical properties to withstand high temperature and have extensive applications to important structures, such as pressure vessels, chemicals plants, aerospace, and pipes and nuclear reactors. Mallik and Kanoria [13], Das and Kanoria [14] applied a periodically varying heat source in generalized thermoelastic functionally graded solid. Abbas [15] discussed the effect of relaxation times in a non-homogeneous hollow cylinder using finite element method. Othman and Abbas [16] studied the generalized thermoelasticity of thermal shock problem in a non-homogeneous isotropic hollow cylinder with energy dissipation. Abbas and Zenkour [17] have constructed a LS model on electro-magneto-thermo-elastic response of an infinite functionally graded cylinder.

The present investigation is devoted to study the fractional order generalized thermoelasticity in a functionally graded material in presence of thermal shock by using Laplace transform and eigenvalue approach. Then the inversion of Laplace transform have been carried out numerically by applying a method of numerical inversion of Laplace transform based on Stehfest technique [18]. Numerical results for all variables in physical space–time domain

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are represented graphically. It is observed that the results of associated Lord and Shulman model [19] and the homogeneous case may easily be recovered from our results by letting the fractional parameter become one and nonhomogeneity parameter become zero respectively.

2. Basic equation

Following Ezzat [11], the basic equations of fractional order theory of thermoelasticity for a functionally graded material in the absence of body forces and heat sources are considered as

The equations of motion

$$\sigma_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{1}$$

The equation of heat conduction

$$e = e_{ii}, \quad i,j = x,y,z, \tag{2}$$

The constitutive equations are given by

$$\sigma_{ij} = 2\mu e_{ij} + [\lambda e - \gamma(T - T_0)]\delta_{ij}, \tag{3}$$

with $e = e_{ii},\ i,\ j = x,\ y,\ z$, where $\lambda,\ \mu$ are the Lame's constants; ρ is the density of the medium; c_e is the specific heat at constant strain; $\gamma = (3\lambda + 2\mu)\alpha_t,\ \alpha_t$ is the coefficient of linear thermal expansion; t is the time; T is the temperature; T_0 is the reference temperature; K is the thermal conductivity; t_0 is the relaxation time;δ_{ij} is the Kronecker symbol;σ_{ij} are the components of stress tensor; u_i are the components of displacement vector. Thus, we replace $\lambda,\ \mu,\ \gamma,\ K$ and ρ by $\lambda_o f(\mathrm{X}),\ \mu_o f(\mathrm{X}),\ \gamma_o f(\mathrm{X}),\ K_o f(\mathrm{X})$ and $\rho_o f(\mathrm{X})$ where $\lambda_o,\ \mu_o,\ \gamma_o,\ K_o$ and ρ_o are assumed to be constants and $f(\mathrm{X})$ is a given dimensionless function of the space variable $\mathrm{X} = (x,y,z)$. Then Eqs. (1)–(3) take the following form

$$f(\mathrm{X})[2\mu_o e_{ij} + [\lambda_o e - \gamma_o(T - T_0)]\delta_{ij}]_{,j} + f(\mathrm{X})_{,j}[2\mu_o e_{ij} + [\lambda_o e - \gamma_o(T - T_0)]\delta_{ij}] = \rho_o f(\mathrm{X})\frac{\partial^2 u_i}{\partial t^2}, \tag{4}$$

$$(K_o f(\mathrm{X})T_{,i})_{,i} = \left(\frac{\partial}{\partial t} + \frac{t_0^{\alpha}}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)(\rho_o c_e f(\mathrm{X})T + \gamma_o f(\mathrm{X})T_0 e), \quad 0 < \alpha \leqslant 1, \tag{5}$$

$$\sigma_{ij} = f(\mathrm{X})[2\mu_o e_{ij} + [\lambda_o e - \gamma_o(T - T_0)]\delta_{ij}], \tag{6}$$

3. Formulation of the problem

Let us consider a functionally graded isotropic thermoelastic body at a uniform reference temperature T_0, occupying the region $x \geqslant 0$ where the x-axis is taken perpendicular to the bounding plane of the half-space pointing inwards. It assumed that the state of the medium depends only on x and the time variable t, so that the displacement vector $\vec{u}$ and temperature field T can be expressed in the following form:

$$\vec{u} = (u(x,t), 0, 0), \quad T = T(x,t). \tag{7}$$

It assumed that the material properties depend only on the x-coordinate. So, we take $f(\mathrm{X})$ as $f(x)$. In the context of the fractional order of generalized thermoelasticity theory based on the Lord and Shulman model, the equation of motion, heat equation, and constitutive equation can be written as

$$f(x)\left[(\lambda_o + 2\mu_o)\frac{\partial^2 u}{\partial x^2} - \gamma_o\frac{\partial T}{\partial x}\right] + \frac{\partial f(x)}{\partial x}\left[(\lambda_o + 2\mu_o)\frac{\partial u}{\partial x} - \gamma_o T\right] = \rho_o f(x)\frac{\partial^2 u}{\partial t^2}, \tag{8}$$

$$K_o f(x)\frac{\partial^2 T}{\partial x^2} + K_o\frac{\partial f(x)}{\partial x}\frac{\partial T}{\partial x} = \left(\frac{\partial}{\partial t} + \frac{t_0^{\alpha}}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\left(\rho_o c_e f(x)T + \gamma_o f(x)T_0\frac{\partial u}{\partial x}\right), \tag{9}$$

$$\sigma_{xx} = f(x)\left[(\lambda_o + 2\mu_o)\frac{\partial u}{\partial x} - \gamma_o(T - T_0)\right], \tag{10}$$

We define the following dimensionless quantities

$$(x', u') = \frac{c}{\chi}(x,u), \quad T' = \frac{T - T}{T_0}, \quad (t', t_0') = \frac{c^2}{\chi}(t, t_0), \quad \sigma_{xx}' = \frac{\sigma_{xx}}{\lambda_o + 2\mu_o}.$$

where $c^2 = \dfrac{\lambda_o + 2\mu_o}{\rho}$ and $\chi = \dfrac{K_o}{\rho_o c_e}$.

Upon introducing in Eqs. (8)–(10), and after suppressing the primes, we obtain

$$f(x)\left[\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial T}{\partial x}\right] + \frac{\partial f(x)}{\partial x}\left[\frac{\partial u}{\partial x} - \beta T\right] = f(x)\frac{\partial^2 u}{\partial t^2}, \tag{11}$$

$$f(x)\frac{\partial^2 T}{\partial x^2} + \frac{\partial f(x)}{\partial x}\frac{\partial T}{\partial x} = \left(\frac{\partial}{\partial t} + \frac{t_0^{\alpha}}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\left(f(x)T + f(x)\varepsilon\frac{\partial u}{\partial x}\right), \tag{12}$$

$$\sigma_{xx} = f(x)\left[\frac{\partial u}{\partial x} - \beta T\right], \tag{13}$$

where $\beta = \frac{T_0\gamma_o}{\lambda_o + 2\mu_o},\ \varepsilon = \frac{\gamma_o}{\rho_o c_e}$, in which $\gamma_o = (3\lambda_o + 2\mu_o)\alpha_t$.

4. Exponential variation of non-homogeneity

We take $f(x) = e^{nx}$, where n is a dimensionless constant [13]. Then Eqs. (11)–(13) reduce to

$$\left[\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial T}{\partial x}\right] + n\left[\frac{\partial u}{\partial x} - \beta T\right] = \frac{\partial^2 u}{\partial t^2}, \tag{14}$$

$$\frac{\partial^2 T}{\partial x^2} + n\frac{\partial T}{\partial x} = \left(\frac{\partial}{\partial t} + \frac{t_0^{\alpha}}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\left(T + \varepsilon\frac{\partial u}{\partial x}\right), \tag{15}$$

$$\sigma_{xx} = e^{nx}\left[\frac{\partial u}{\partial x} - \beta T\right]. \tag{16}$$

5. Application

In order to solve the problem, both the initial conditions and the boundary conditions needed to be considered. The initial conditions of the problem are assumed to be homogeneous. Then we have

$$u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0, \quad T(x,0) = \frac{\partial T(x,0)}{\partial t} = 0. \tag{17}$$

We consider the problem of a thick plate of finite high l. Choosing the x-axis perpendicular to the surface of the plate with the origin coinciding with the lower plate, the region Ψ under consideration becomes:

Ψ is function of $\{(x,y,x) : 0 \leqslant x \leqslant l,\ -\infty < y < \infty,\ -\infty < z < \infty\}$. The surface of the plate is taken to be traction free. The lower plate is subjected to a thermal shock. The upper plate is kept at zero temperature. Mathematically these can be written

$$\sigma_{xx}(0,t) = 0, \quad T(0,t) = T_1 H(t), \tag{18}$$

$$\sigma_{xx}(l,t) = 0, \quad T(l,t) = 0, \tag{19}$$

where $H(t)$ denotes the Heaviside unit step function.

6. Governing equations in the Laplace transform domain

Applying the Laplace transform for Eqs. (14)–(19) define by the formula