Probabilistic Engineering Mechanics 34 (2013) 131–156

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Stiffness tensor random fields through upscaling of planar random materials

Michael P. Sena [a], Martin Ostoja-Starzewski [a,*], Luis Costa [b]

[a] Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, 1206 W. Green St., Urbana, IL 61801, USA
[b] Institute for Multiscale Reactive Modeling Energetics and Warheads Research and Development RDAR-MEE-W, Building 3022, Room 44 Picatinny, NJ 07806-5000, USA

ARTICLE INFO

Article history:
Received 16 October 2012
Received in revised form
11 July 2013
Accepted 26 August 2013
Available online 20 September 2013

Keywords:
Random microstructure
Random fields
Stiffness tensor
Compliance tensor
Mesoscale
Correlation structure

ABSTRACT

Unique effective material properties are not possible for random heterogeneous materials at intermediate length scales, which is to say at some *mesoscale* above the microscale yet prior to the attainment of the representative volume element (RVE). Focusing on elastic moduli in particular, a micromechanical analysis based on the Hill–Mandel condition leads to the conclusion that two fields, stiffness and compliance, are required to bound the response of the material. In particular, we analyze means and correlation coefficients of a random planar material with a two-phase microstructure of random checkerboard type. We employ micromechanics, which can be viewed as an upscaling, smoothing procedure using the concept of a mesoscale "window", and random field theory to compute the correlation structure of 4th-rank tensor fields of stiffness and compliance for a given mesoscale. Results are presented for various correlation distances, volume fractions, and contrasts in stiffness between phases. The main contribution of this research is to provide the data for developing analytical correlation functions, which can then be used at any mesoscale to generate micromechanically based inputs into analytical and computational mechanics models.

1. Introduction

The assumption of constitutive relations is central to the theory and application of continuum mechanics. Within the linear elastic theory for solid materials, the generalized version of Hooke's law utilizes tensors of elastic moduli to establish the appropriate relationship between stress and strain. Boundary value problems on a scale of, say, 1 m can be solved, even in the presence of material spatial fluctuations, because of material tensor-valued moduli linking stresses and strains. If this is done by, say, a finite element (FE) method with each element of size, say, 2 mm, the key challenge is to describe those fluctuations in a manner consistent with micromechanics. That is, the upscaling from the underlying random microstructure—i.e., the homogenization on the scale of 2 mm—has to be done right. If one is interested in elastic modulus on a scale of 4 mm, however, the stiffness tensor random field (RF) will be different (it will, at the very least, exhibit weaker fluctuations), and this again has to be done right.

In general, there is no "separation of length scales" between the microscale and the macroscale, and we must consider the *mesoscale*, be it 2 mm or 4 mm in our example above. In other words, the stiffness field is scale dependent. With this challenge in mind, we follow the methodology developed earlier for characterization of mesoscale thermal conductivity tensor fields in planar random media (mathematically equivalent to stiffness field in anti-plane elasticity of media having the same random geometries) [1–3] to assess the in-plane stiffness tensor RF. Once this is done, fields like these, based on the microstructure of a specific material, should inform analytical models of stiffness tensor fields, such as based on the maximum entropy approach [4–9], so as to form input into stochastic continuum models (e.g., wavefronts in random media) or be incorporated into computer codes in order to generate microstructurally based stiffness matrices in stochastic finite element (SFE) methods.

The geometries of underlying material microstructures studied here are those of a planar Bernoulli lattice process at various values of probability p, where p is the event at the lattice site. The event signifies a stiff phase relative to a matrix phase. The nominal volume fraction of the stiff phase is directly given by p. We restrict our scope to a random two-phase composite material in the setting of in-plane linear (hyper-)elasticity, specifically, assumed to be in a state of plane strain. We covered a wide range of contrasts (mismatches between the phases) in the material: up to 1000.

To perform upscaling, RFs on coarser length scales (mesoscales) are specified by their one-point and two-point statistics. However,

* Corresponding author. Tel.: +1 217 265 0900; fax: +1 217 244 9980.
E-mail address: martinos@illinois.edu (M. Ostoja-Starzewski).

0266-8920/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.probengmech.2013.08.008

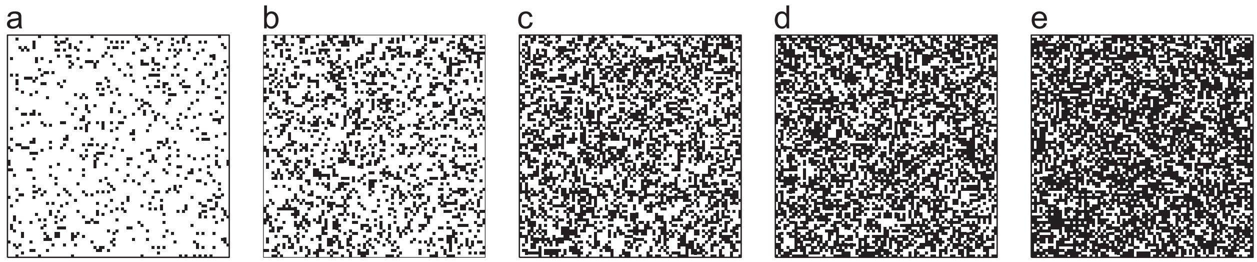

Fig. 1. Realizations of the binomial point process (random checkerboard) at five different p's. (a) 10%, (b) 25%, (c) 41%, (d) 50% and (e) 59%.

starting from the assumption that spatial correlations of these fields are neither intuitive nor derivable through analytical methods, we employ a Monte Carlo approach—that is, repeated sampling of a random material—to estimate these statistics and reveal the general structure of the fields. The technique adheres to the Hill–Mandel condition of micromechanics so as to ensure that the upscaling ensures agreement between theory and experiment. While the individual phases on the microscale are isotropic, the mesoscale domains exhibit anisotropy with probability one. Also, since the mesoscale windows are statistical volume elements of tensor-valued RFs of stiffness, all the components of the in-plane stiffness tensor have to be assessed and, likewise, all their auto- and cross-correlations.

While the Bernoulli lattice process offers a very simple random medium model, we ask whether the conclusions concerning the upscaling of even this microstructural model are obvious or not. If not, this study should provide guidance as to what may be expected in richer models. In brief, the contents of this paper are as follows: (i) background introduction of a random material model and upscaling; (ii) mesoscale RFs of stiffness and compliance for in-plane linear elasticity: basic concepts and computational methods used to determine spatial correlation structures, followed by extensive numerical results; (iii) conclusions.

2. Background

2.1. Random material

We consider for a random microstructure model the well-known Bernoulli lattice process in plane, with each lattice node being occupied by either one of the two phases. To make things clear, consider a Cartesian lattice of spacing a in $\mathbb{R}^2$, that is

$$L_a = \{\mathbf{x} = (ma, ma)\}, \tag{1}$$

where m is an integer. We recall that a Bernoulli lattice process $\Phi_{p,a}$ on L_a is a random subset of the lattice where each point of L_a is contained in $\Phi_{p,a}$ with probability p independent of all the other points. If the random variable $\Phi_{p,a}(A)$ is the number of points in A, then it is binomially distributed with parameters $p$and n (the number of lattice points that belong to A). If $A_1, A_2, \ldots, A_k$ are pairwise disjoints, then $\Phi_{p,a}(A_1), \Phi_{p,a}(A_2), \ldots, \Phi_{p,a}(A_k)$ are independent and

$$P\{\Phi_{p,a}(A) = k\} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n. \tag{2}$$

The simulation of a Bernoulli lattice process proceeds along simple lines: for each point $\mathbf{x} \in L_a$, generate a random variable z_k, uniformly distributed in [0, 1], and accept this point if $z_k < p$. Then, $\Phi_{p,a}$ is the union of all such points; five examples for different p's are shown in Fig. 1. Note that p directly plays the role of the nominal volume fraction, which we denote $v_f^{(s)}$, where s stands for the stiff phase as opposed to c for the compliant phase. The cases

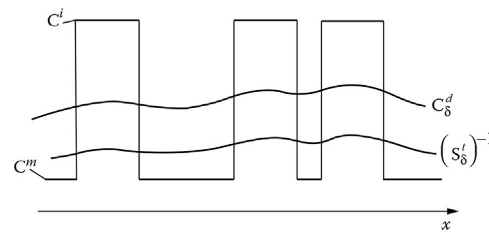

Fig. 2. The setup of random fields: from a piecewise-constant realization of a composite to two approximating continua at a finite mesoscale. Here the superscript ^{d} stands for essential (i.e., displacement ^{d}), while ^{n} for natural (i.e., traction ^{t}) boundary conditions.

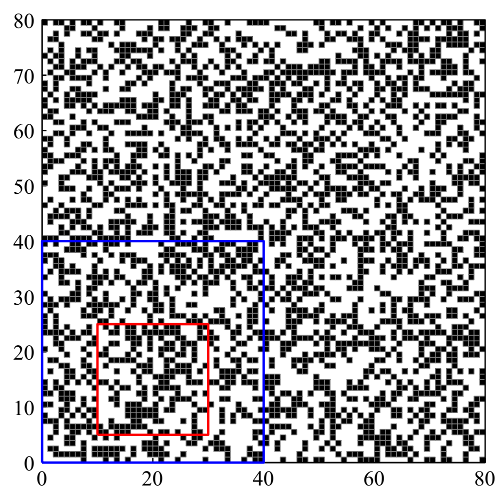

Fig. 3. Mesoscale window (red frame) being shifted in both x_1 and x_2 within a given range (blue frame) on a single realization of the random checkerboard microstructure. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

of $p = 0.41$ and 0.59 correspond to site percolations on planar lattices, but, since the response of a soft matrix with stiff inclusions cannot be mapped into that of a stiff matrix with soft inclusions on finite scales [3], both systems have to be considered separately.

A note on notation: we use $\mathbf{A}, \mathbf{B}, \ldots$ to denote a tensor symbolically, and $A_i, B_{ij}, \ldots$ for a subscript notation of tensors of 1st-rank, 2nd-rank, and so on…; a comma is used to indicate partial differentiation.

In effect, we have the random material $\mathcal{B} = \{B(\omega); \omega \in \Omega\}$ given as a random checkerboard where each $B(\omega)$ is one realization whose mechanics is deterministic. Each realization of the random checkerboard $B(\omega)$ is assumed to be piecewise constant, consisting entirely of perfectly bonded, isotropic phases.