Probabilistic Engineering Mechanics 34 (2013) 168–176

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Solution of linear dynamic systems with uncertain properties by stochastic reduced order models

M. Grigoriu

Cornell University, Ithaca, NY 14853-3501, USA

ARTICLE INFO

Article history:
Received 7 January 2013
Accepted 6 September 2013
Available online 25 September 2013

Keywords:
Linear random vibration
Probability theory
Random vectors
Stochastic reduced order models (SROMs)
Systems with uncertain properties

ABSTRACT

A novel method, referred to as the stochastic reduced order model (SROM) method, is proposed for finding statistics of the state of linear dynamic systems with random properties subjected to random noise. The method is conceptually simple, accurate, computationally efficient, and non-intrusive in the sense that it uses existing solvers for deterministic differential equations to find state properties.

Bounds are developed on the discrepancy between the exact and the SROM solutions under some assumptions on system properties. The bounds show that the SROM solutions converge to the exact solutions as the SROM representation of the vector of random system parameters is refined. Numerical examples are presented to illustrate the implementation of the SROM method and demonstrate its accuracy and efficiency.

1. Introduction

Monte Carlo simulation is the only general method for estimating state statistics for linear/nonlinear dynamic systems with uncertain properties subjected to random noise. Computational time, which is usual excessive when dealing with realistic systems, limits the usefulness of the method. Alternative methods have been proposed for solving this class of random vibration problems approximately, for example, conditional analysis, stochastic Galerkin and collocation, state augmentation, and Taylor, perturbation, and Neumann series. An extensive review of features and limitations of these methods can be found in [6, Sections 7.3–7.5]. Under some assumptions on system properties and their uncertain parameters, these methods have been used successfully in applications [4,5,10,12,13,15,18].

The proposed SROM method provides an additional tool for calculating approximately state statistics for dynamic systems with uncertain properties subjected to random noise. The method can be used to solve both linear and nonlinear systems, but this study deals only with linear systems. Let Z be an $\mathbb{R}^q-$valued random variable characterizing the uncertain properties of a dynamic system and let $X(t)$ be an $\mathbb{R}^d-$valued stochastic process denoting the system state. Properties of the conditional process $X(t)|Z$ can be obtained by methods of linear random vibration. This observation can be used to construct approximations for statistics of the unconditional state $X(t)$.

Let $\tilde{Z}$ be a SROM for Z, that is, an $\mathbb{R}^q-$valued random variable with a finite number of samples $\{\tilde{z}_k\}$ of probabilities $\{p_k\}$, $k=1,\ldots,m$, that are selected from the samples of Z such that $\tilde{Z}$ and Z have similar properties. The proposed SROM method uses properties of the conditional processes $\{X(t)|Z=\tilde{z}_k\}$ to construct surrogate models $\tilde{X}(t)$ for $X(t)$. The performance of the SROM method is assessed by bounds on the discrepancy $X(t)-\tilde{X}(t)$ between exact and SROM solutions. It is shown that the SROM solution $\tilde{X}(t)$ is (1) guaranteed to converge to $X(t)$ as the SROM representation of Z is refined, (2) computational efficient since it requires a limited exploration of the range of Z that is guided by SROMs $\tilde{Z}$ of Z, and (3) non-intrusive since its implementation uses existing solvers for deterministic differential equations.

Two numerical examples are used to illustrate the method implementation and demonstrate its accuracy. The first is a half oscillator subjected to Gaussian white noise whose properties are modeled by a real-valued random variable Z. The second is a three degree of freedom system subjected to a first order stationary Gaussian process with exponential correlation function. The Monte Carlo and the SROM methods are used to estimate the probability that a vector system response leaves a safe set during a time interval, referred to as failure probability. The performance of the SROM method is remarkable in both example even when based in SROMs $\tilde{Z}$ with only $m=5$ samples.

2. Problem statement

Suppose $X(t)$ is the solution of the linear differential equation

$$\dot{X}(t)=a(t,Z)X(t)+Y(t), \quad t\geq t_0, \tag{1}$$

where $a(t,Z)$ is an (d,d)-matrix that depends on time and the q-dimensional random vector Z of uncertain system parameters and $Y(t)$ is an $\mathbb{R}^d-$valued stochastic process defining the input. It is

E-mail address: mdg12@cornell.edu

0266-8920/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.probengmech.2013.09.001

assumed that (1) Z and $Y(t)$ are independent random elements that may be defined on distinct probability spaces $(\Omega, \mathcal{F}, P)$ and $(\Omega', \mathcal{F}', P')$, (2) the probability laws of Z and $Y(t)$ are known, and (3) the initial state $X(t_0)$, if random, is independent of Z and $Y(t)$. As previously stated, conditional on Z, Eq. (1) defines a classical linear random vibration problem. The unconditional state $X(t)$ is a stochastic process defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', P \times P')$. To emphasize the dependence of the system state on Z and initial time t_0, the system state $X(t)$ will be denoted at times by $X(t, t_0, Z)$.

The solution of Eq. (1) can be given in the form

$$X(t, t_0, Z) = \varphi(t, t_0, Z)X(t_0) + \int_{t_0}^{t} \varphi(t, s, Z)Y(s)\, ds, \quad t \geq t_0, \tag{2}$$

where the (d,d)-transition matrix $\varphi(\cdot, t_0, Z)$ is the solution of

$$\dot{\varphi}(t, t_0, Z) = a(t, Z)\varphi(t, t_0, Z), \quad t \geq t_0, \tag{3}$$

with $\varphi(t_0, t_0, Z) = I_d$, where I_d is the (d,d)-identity matrix and the dot denotes differentiation with respect to time [2, Sections 3 and 4].

Methods are available for finding properties of $X(t, t_0, Z)$ conditional on Z. If $Y(t)$ has finite variance, the first two moments of $X(t, t_0, Z)|Z$ satisfy ordinary differential equations available in the linear random vibration literature [3,14]. If $Y(t)$ is a Gaussian process, the first two moments of $X(t, t_0, Z)|Z$ define completely the distribution of this conditional process. Otherwise, methods of stochastic differential equation need to be used for solution. If $Y(t)$ is a polynomial in linear diffusion processes, Itô's formula can be applied to develop differential equations for moments of any order of $X(t, t_0, Z)|Z$ [6, Section 7.2.2]. If $Y(t)$ does not have finite variance, partial differential equations can be developed for the characteristic function of $X(t, t_0, Z)|Z$ based on properties of diffusion processes and stochastic integrals [6, Section 7.2.3].

Suppose properties of the conditional process $X(t, t_0, Z)|Z$ have been obtained. Unconditional state statistics can be calculated by, for example, Monte Carlo simulation from properties of $\{X(t, t_0, Z)|Z = z_i\}$, where $\{z_i\}$, $i = 1,\ldots,n$, are independent samples of Z. If $Y(t)$ has finite variance, the conditional mean and covariance functions $\mu(t, Z) = E[X(t, t_0, Z)|Z]$ and $r(s, t, Z) = E[X(s, t_0, Z)X(t, t_0, Z)|Z]$ exist so that the corresponding functions for the unconditional state can be estimated by

$$\hat{\mu}(t) = \frac{1}{n}\sum_{i=1}^{n} \mu(t, z_i) \quad \text{and} \quad \hat{r}(s, t) = \frac{1}{n}\sum_{i=1}^{n} r(s, t, z_i). \tag{4}$$

3. SROM method

The implementation of the SROM method involves three steps. First, a SROM $\tilde{Z}$ is developed for Z, that is, a random vector that approximates Z in some sense and has a finite number of samples $\{\tilde{z}_k\}$, $k = 1,\ldots,m$, selected from the samples of Z. Second, properties of conditional state processes $\{\tilde{X}_k(t, t_0) = X(t, t_0, Z)|Z = \tilde{z}_k\}$ and their gradients $\{\nabla\tilde{X}_k(t, t_0) = (\partial X(t, t_0, Z)/\partial Z_1, \ldots, \partial X(t, t_0, Z)/\partial Z_q)|Z = \tilde{z}_k\}$ are calculated. Third, the processes $\{\tilde{X}_k(t, t_0)\}$ and $\{\nabla\tilde{X}_k(t, t_0)\}$ are used to construct an approximation $\tilde{X}(t, t_0, Z)$ for the unconditional state vector $X(t, t_0, Z)$, that constitutes a surrogate model for the system state.

The SROM method is closely related to Monte Carlo simulation, and can be viewed as a smart Monte Carlo simulation. Like Monte Carlo simulation, the SROM method uses properties of the conditional state $X(t, t_0, Z)|Z$ corresponding to samples of Z. In contrast to Monte Carlo that uses a large number of equally likely samples of Z selected at random, the SROM method uses a few samples of Z that are selected in an optimal manner, that is, the samples of SROMs $\tilde{Z}$ for Z. Moreover, the SROM-based surrogate model $\tilde{X}(t, t_0, Z)$ also accounts for the local variation of the system state with Z in the vicinities of the samples $\{\tilde{Z}_k\}$ of $\tilde{Z}$.

3.1. SROMs for random vectors

Details on the construction and properties of SROMs can be found in [9,11,17]. This section outlines briefly the essentials of SROMs for the special case in which Z is a real-valued random variable ($q = 1$). Let $\{\tilde{z}_k\}$, $k = 1,\ldots,m$, be a set of size m selected from samples of Z. Denote by $\{\Gamma_k\}$ the cells of a Voronoi tessellation constructed in the range $\Gamma = Z(\Omega)$ of Z with centers $\{\tilde{z}_k\}$, that is, $\Gamma_k = \{z \in \Gamma : \|z - \tilde{z}_k\| < \|z - \tilde{z}_l\|, l \neq k\}$. If $z \in \Gamma$ such that $\|z - \tilde{z}_k\| = \|z - \tilde{z}_l\|$, $l \neq k$, then z can be allocated to either Γ_k or Γ_l. The Voronoi cells $\{\Gamma_k\}$ are intervals for $q = 1$.

The probability law of a SROM $\tilde{Z}$ of Z is completely defined by its samples $\{\tilde{z}_k\}$ and their probabilities $\{p_k = P(Z \in \Gamma_k)\}$. For example, moments of order r and the distribution of $\tilde{Z}$ are $\tilde{\mu}(r) = E[\tilde{Z}^r] = \sum_{k=1}^{m} p_k \tilde{z}_k^r$ and $\tilde{F}(z) = \sum_{k=1}^{m} p_k 1(z \geq \tilde{z}_k)$, respectively. Our objective is to identify the pairs $(\tilde{z}_k, \Gamma_k)$, $k = 1,\ldots,m$, that minimize the discrepancy between properties of $\tilde{Z}$ and Z for a selected model size m. This discrepancy can be measured by objective functions of the type $\sum_{r=1}^{\bar{r}} w_r(\tilde{\mu}(r) - \mu(r))^2 + \int (\tilde{F}(z) - F(z))^2 \beta(z)\, dz$, where $w_r, \beta(\cdot) > 0$ are weighting coefficients, $\bar{r} \geq 1$ is an arbitrary integer, $\mu(r) = E[Z^r]$, and $F(z) = P(Z \leq z)$. Algorithms have been developed for constructing SROMs that are optimal in the sense that their defining parameters $(\tilde{z}_k, \Gamma_k)$, $k = 1,\ldots,m$, are such that a prescribed objective function is minimized [17].

3.2. Proposed SROM solution

Let $\tilde{Z}$ be a SROM for Z with samples $\{\tilde{z}_k\}$ of probabilities $\{p_k\}$ corresponding to Voronoi cells $\{\Gamma_k\}$, $k = 1,\ldots,m$, partitioning the range $\Gamma = Z(\Omega)$ of Z. A SROM with m samples is said to have size m. As previously, $\{\tilde{X}_k(t, t_0)\}$ and $\{\nabla\tilde{X}_k(t, t_0)\}$ are the conditional states $\{X(t, t_0, Z)|Z = \tilde{z}_k\}$ and gradients $\{(\partial X(t, t_0, Z)/\partial Z_1, \ldots, \partial X(t, t_0, Z)/\partial Z_q)|Z = \tilde{z}_k\}$, $k = 1,\ldots,m$. The SROM solution $\tilde{X}(t, t_0, Z)$ has the expression

$$\tilde{X}(t, t_0, Z) = \sum_{k=1}^{m} 1(Z \in \Gamma_k)[\tilde{X}_k(t, t_0) + \nabla\tilde{X}_k(t, t_0) \cdot (Z - \tilde{z}_k)], \quad t \geq t_0, \tag{5}$$

so that its implementation requires knowledge of $\{\tilde{X}_k(t, t_0)\}$ and $\{\nabla\tilde{X}_k(t, t_0)\}$, that is, system states and sensitivity factors conditional on m values of Z, the samples of $\tilde{Z}$. As previously stated, $\tilde{X}(t, t_0, Z)$ is viewed as a surrogate model for the unconditional system state $X(t, t_0, Z)$.

We conclude this section with three comments on SROM solutions. First, $X(t, t_0, Z)$ constitutes a response surface on the range Γ of Z for any time t and sample of $Y(t)$. The SROM solution $\tilde{X}(t, t_0, Z)$ in Eq. (5) is a piecewise linear approximation of this response surface consisting of hyperplanes tangent to it at the centers $\{\tilde{z}_k\}$ of the Voronoi cells $\{\Gamma_k\}$. Second, the distribution of Z guides the construction of the partition $\{\Gamma_k\}$ of Γ, in construct to, for example, the stochastic collocation method that interpolates in Γ with respect to points whose selection is independent on the distribution of Z [6, Section 7.4.8]. Third, the generation of samples of $\tilde{X}(t, t_0, Z)$ involves elementary calculations and is efficient. It does not require to construct the Voronoi tessellation $\{\Gamma_k\}$; a sample z_i of Z is allocated to a particular Voronoi cell based on its distances $\|z_i - \tilde{z}_k\|$ to the centers of these cells.

4. Accuracy of SROM method

Consider the model in Eq. (5) with $m = 1$, that is, the model

$$\tilde{X}(t, t_0, Z) = \tilde{X}(t, t_0) + \nabla\tilde{X}(t, t_0) \cdot (Z - \tilde{z}), \quad t \geq t_0, \tag{6}$$