Probabilistic Engineering Mechanics 34 (2013) 200–210

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

# A two-step method for analysis of linear systems with uncertain parameters driven by Gaussian noise

M. Grigoriu [a], R.V. Field Jr. [b,*,1]

[a] Cornell University, Ithaca, NY 14853, USA
[b] Sandia National Laboratories, Albuquerque, NM 87185-0346, USA

ARTICLE INFO

*Article history:*
Received 29 January 2013
Received in revised form
9 October 2013
Accepted 10 October 2013
Available online 18 October 2013

*Keywords:*
Bayesian analysis
Monte Carlo simulation
Random vibration
Stochastic differential equations
Stochastic reduced order models

ABSTRACT

A two-step method is proposed to find state properties for linear dynamic systems driven by Gaussian noise with uncertain parameters modeled as a random vector with known probability distribution. First, equations of linear random vibration are used to find the probability law of the state of a system with uncertain parameters conditional on this vector. Second, stochastic reduced order models (SROMs) are employed to calculate properties of the unconditional system state. Bayesian methods are applied to extend the proposed approach to the case when the probability law of the random vector is not available. Various examples are provided to demonstrate the usefulness of the method, including the random vibration response of a spacecraft with uncertain damping model.



## 1. Introduction

Consider a linear dynamic system with uncertain properties that can be described by a random vector, and suppose the system is driven by Gaussian noise. This scenario is consistent with, for example, linear random vibration analysis of systems with uncertain constitutive properties, also referred to as "disordered structures." [1, Chapter 9]. In addition to Monte Carlo simulation, there are two methods for determining the probability law of the system state, provided the distribution of the random vector characterizing the uncertain system properties is known. One approach is to augment the system state vector with new states corresponding to the uncertain parameters. The added states are time invariant, so that their time derivatives are zero, and the entire augmented state vector satisfies a nonlinear system of equations driven by Gaussian noise [2, Section 9.2.4]. Monte Carlo simulation is the only general method for solving this class of stochastic equations, that is, nonlinear random vibration problems. However, the Monte Carlo approach is inefficient when dealing with large dimensional systems, that is, systems commonly encountered in applications.

The second method, referred to as conditional analysis, exploits the fact that the state vector of a linear system with uncertain parameters driven by Gaussian noise is a Gaussian process for specified values of these parameters. Properties of the unconditional state vector can be obtained from those of its conditional properties by, for example, Monte Carlo simulation [3,4]. This approach is conceptually simple, but is impractical when dealing with complex systems that depend on a large number of uncertain parameters.

This study proposes an alternative method for calculating state properties for linear dynamic systems with uncertain parameters driven by Gaussian noise. The proposed approach utilizes stochastic reduced order models (SROMs) [5,6], rather than traditional Monte Carlo simulation, to calculate properties of the system state vector from its conditional statistics. The computational advantage of the method relative to Monte Carlo simulation is significant because it involves a limited exploration of the space of uncertain parameters. The proposed method is straightforward when the probability law of the random vector modeling all uncertain variables is known. However, a modification of the method is also developed herein to consider the more interesting case where the probability law of uncertain parameters is unknown.

The outline of the paper is as follows. In Section 2, we review the second-moment properties of deterministic linear dynamic systems driven by random noise, discuss briefly the construction of SROMs, and illustrate by example the implementation of the proposed method for finding state properties for linear systems with uncertain properties in a Gaussian environment. To illustrate

* Corresponding author. Tel.: +1 505 284 4060; fax: +1 505 844 6620.
*E-mail addresses:* mdg12@cornell.edu (M. Grigoriu), rvfield@sandia.gov (R.V. Field Jr.).

[1] Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under Contract DE-AC04-94AL85000.

0266-8920/$ - see front matter 
http://dx.doi.org/10.1016/j.probengmech.2013.10.003

the proposed method on a problem of engineering interest, we also consider the random vibration response of a spacecraft during atmospheric re-entry, where the parameters describing the damping of the structure are uncertain. In this section it is assumed that the probability law of the uncertain parameters is known. In Section 3, it is assumed that the probability law of the uncertain parameters is not known. Two cases are examined. First, it is assumed that the uncertain parameters can be measured. Second, it is assumed that the system parameters are not observable directly. Estimates are constructed for the distribution of the uncertain system parameters within a Bayesian framework. Once these estimates are obtained, the method outlined in the previous section can be used to find unconditional state properties. Numerous examples are used to illustrate the proposed method.

## 2. Linear systems with uncertain parameters

Let $\mathbf{X}(t)$ be a $\mathbb{R}^d$–valued stochastic process denoting the state of a linear system that depends on a finite number of uncertain properties and is subjected to Gaussian noise. The uncertain parameters of the system are collected in a $q$-dimensional random vector $\mathbf{\Theta}$. The conditional state $\mathbf{X}(t)|\mathbf{\Theta}$ is a Gaussian process, so that the probability law of this process is completely specified by its mean and covariance functions. The first two moments of $\mathbf{X}(t)|\mathbf{\Theta}$ satisfy ordinary differential equations for specified values of $\mathbf{\Theta}$ that, generally, need to be solved numerically. Analytical solutions for the second moment properties of $\mathbf{X}(t)|\mathbf{\Theta}$ are available in special cases of limited practical interest.

Our objective is to develop an efficient and accurate method for calculating properties of the unconditional system state $\mathbf{X}(t)$ and of functionals of this process. The method involves two steps. First, the second moment properties of the conditional state $\mathbf{X}(t)|\mathbf{\Theta}$ are calculated, so that the law of the Gaussian process $\mathbf{X}(t)|\mathbf{\Theta}$ is completely defined. Second, properties of functionals of $\mathbf{X}(t)$ are derived from the probability law of $\mathbf{X}(t)|\mathbf{\Theta}$ and the distribution of $\mathbf{\Theta}$. We assume in this section that the distribution of $\mathbf{\Theta}$ is known.

**Example 1.** Let $X(t)$, $t \geq 0$, be the solution of the stochastic differential equation

$$\mathrm{d}X(t) = -\Theta\, X(t)\,\mathrm{d}t + \sqrt{2\Theta}\,\mathrm{d}B(t), \quad t \geq 0, \tag{1}$$

where parameter $\Theta > 0$ is a real-valued random variable, $B(t)$ denotes a Brownian motion, and the initial state $X(0) \sim N(0,1)$ is independent of $B(t)$. The mean and variance equations for $X(t)|\Theta$ are well known and equal to $\dot{\mu}(t) = -\Theta\mu(t)$ and $\dot{\gamma}(t) = -2\Theta\gamma(t) + 2\Theta$, respectively, with initial conditions $\mu(0) = 0$ and $\gamma(0) = 1$, so that $\mu(t) = 0$ and $\gamma(t) = 1$ at all times $t \geq 0$. Further, the covariance function $c(s) = \mathrm{E}[X(t+s)X(t)|\Theta]$ satisfies the differential equation $\mathrm{d}c(s)/\mathrm{d}s = -\Theta c(s)$ with $c(0) = 1$, so that $X(t)|\Theta$ is a stationary Gaussian process with zero mean and covariance function $c(s) = \exp(-\Theta|s|)$. Since $X(t)|\Theta$ is stationary, its second moment properties can be calculated more efficiently by analysis in the frequency domain. For example, the spectral density of this process is $s(\nu) = \Theta/(\pi(\nu^2 + \pi^2))$.

The finite dimensional distributions of $X(t)$, that is, the unconditional state of Eq. (1), can be calculated from

$$P(X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n) = \mathrm{E}_\Theta[\Phi_n(x_1, \ldots, x_n; \rho(\Theta))], \tag{2}$$

where $n \geq 1$ is an integer, $t_1, \ldots, t_n$ are arbitrary times, $\mathrm{E}_\Theta$ denotes expectation with respect to $\Theta$, and $\Phi_n$ is the joint distribution of an $n$-dimensional Gaussian vector with mean zero and covariance matrix $\rho(\Theta) = \{\exp(-\Theta\,|t_i - t_j|, i,j = 1, \ldots, n\}$. The finite dimensional distributions of $X(t)$ can be calculated efficiently in this case by Monte Carlo simulation since $X(t)$ is a real-valued process and $\Theta$ a random variable. However, the construction of Monte Carlo estimates for properties of functionals of $X(t)$ becomes impractical in realistic problems dealing with high dimensional systems depending on many uncertain parameters since this requires the generation of many independent samples of $X(t)|\Theta$ for each realization of $\Theta$. □

**Example 2.** Let $X(t)$ be the process defined by Eq. (1), and consider a particular functional of $X(t)$ defined by $X_\tau = \max_{0 \leq t \leq \tau}\{X(t)\}$, $0 < \tau < \infty$. Our objective is to construct an estimator for the distribution of $X_\tau$. Consider the conditional estimator

$$\hat{F}_\tau(\xi|\theta) = \frac{1}{b}\sum_{i=1}^{b} 1(X_{\tau,i} \leq \xi|\Theta = \theta), \tag{3}$$

where $b \geq 1$ is an integer, $1(A)$ is an indicator function equal to one if event $A$ is true and zero otherwise, and $\{X_{\tau,i}, i = 1, \ldots, b\}$ denote independent copies of $X_\tau|(\Theta = \theta)$. This conditional estimator is unbiased, that is, $\mathrm{E}[\hat{F}_\tau(\xi|\theta)] = P(X_\tau \leq \xi|\Theta = \theta)$, with variance

$$\mathrm{Var}[\hat{F}_\tau(\xi|\theta)] = \frac{1}{b}P(X_\tau \leq \xi|\Theta = \theta)(1 - P(X_\tau \leq \xi|\Theta = \theta)), \tag{4}$$

that converges to 0 as $b \to \infty$. Since, for $\varepsilon > 0$ arbitrary, we have

$$P(|\hat{F}_\tau(\xi|\theta) - P(X_\tau \leq \xi|\Theta = \theta)| > \varepsilon) \leq \frac{\mathrm{Var}[\hat{F}_\tau(\xi|\theta)]}{\varepsilon^2} \tag{5}$$

by Chebyshev's inequality [2, Section 2.12], the discrepancy between $\hat{F}_\tau(\xi|\theta)$ and $P(X_\tau \leq \xi|\Theta = \theta)$ can be made as small as desired by increasing $b$.

The unconditional estimator for the distribution of $X_\tau$ is

$$\hat{F}_\tau(\xi) = \mathrm{E}_\Theta[\hat{F}_\tau(\xi|\Theta)] = \int \hat{F}_\tau(\xi|\theta) f(\theta)\,\mathrm{d}\theta, \tag{6}$$

where $f(\theta)$ denotes the probability density of random variable $\Theta$ defined by Eq. (1). Generally, the integral in Eq. (6) needs to be calculated numerically. The calculation of this integral by numerical integration or Monte Carlo simulation may not be feasible since these methods require estimators $\hat{F}_\tau(\xi|\theta)$ for many values of $\Theta$, and the construction of each conditional estimator is based on generated samples of $X(t)$ for specified values of $\Theta$. We propose an alternative approach, that is based on the representation of $\Theta$ by stochastic reduced order models (SROMs). □

### 2.1. SROMs for uncertain system parameters

A SROM $\tilde{\mathbf{\Theta}}$ for $\mathbf{\Theta} \in \mathbb{R}^q$ is a simple random vector with samples $\tilde{\boldsymbol{\theta}}_k \in \mathbb{R}^q$ of probabilities $p_k$, $k = 1, \ldots, m$, where $m \geq 1$ is an integer, $p_k \geq 0$, and $\sum_{k=1}^{m} p_k = 1$ [5,6]. The random variable $\tilde{\mathbf{\Theta}}$ is defined on the same probability space as $\mathbf{\Theta}$. Any collection $(\tilde{\boldsymbol{\theta}}_k, p_k)$, $k = 1, \ldots, m$, of samples and probabilities defines a SROM $\tilde{\mathbf{\Theta}}$ for $\mathbf{\Theta}$. We are interested in a stochastic reduced order model $\tilde{\mathbf{\Theta}}$ that, for a selected size $m$, describes the probability law of $\mathbf{\Theta}$ in an optimal sense.

Optimization algorithms have been proposed to construct SROMs [5]. For example, if $\mathbf{\Theta} = \Theta$ is a real-valued random variable, the optimal probabilities $\{p_k\}$ for a set $(\tilde{\theta}_1, \ldots, \tilde{\theta}_m)$ of selected samples can be obtained by minimizing the discrepancy between the distributions and some of the moments of $\Theta$ and $\tilde{\Theta}$. This discrepancy can be quantified by the objective function

$$e(p_1, \ldots, p_m) = \alpha_1 \max_\theta |F(\theta) - \tilde{F}(\theta)| + \alpha_2 \sum_{r=1}^{\overline{r}} \left| \frac{\mu(r) - \tilde{\mu}(r)}{\mu(r)} \right|, \tag{7}$$

where $F$ denotes the distribution of $\theta$, $\mu(r) = \mathrm{E}[\theta^r]$, $\tilde{F}(\theta) = \sum_{k=1}^{m} p_k 1(\tilde{\theta}_k \leq \theta)$ denotes the distribution of $\tilde{\theta}$, $\tilde{\mu}(r) = \mathrm{E}[\tilde{\theta}^r] = \sum_{k=1}^{m} p_k \tilde{\theta}_k^r$, $\overline{r} \geq 1$ is an integer, and $\alpha_1, \alpha_2 > 0$ are coefficients selected such that the two terms of the objective function $e(p_1, \ldots, p_m)$ have similar order of magnitude.