Probabilistic Engineering Mechanics 25 (2010) 61–66

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Reliability of quasi integrable generalized Hamiltonian systems

L.C. Chen, W.Q. Zhu *

Department of Mechanics, State Key Laboratory of Fluid Power Transmission and Control, Zhejiang University, Hangzhou 310027, China

ARTICLE INFO

Article history:
Received 10 October 2008
Accepted 8 June 2009
Available online 11 June 2009

Keywords:
Quasi-integrable generalized Hamiltonian system
First passage failure
Stochastic averaging

ABSTRACT

The reliability of quasi-integrable generalized Hamiltonian systems is studied. An m-dimensional integrable generalized Hamiltonian system has M Casimir functions $C_1, \ldots, C_M$ and n ($n = (m - M)/2$) independent first integrals $H_{M+1}, \ldots, H_{M+n}$ in involution. When an integrable generalized Hamiltonian system is subjected to light dampings and weakly stochastic excitations, it becomes a quasi-integrable generalized Hamiltonian system. The averaged Itô equations for slowly processes $C_1, \ldots, C_M, H_{M+1}, \ldots, H_{M+n}$ can be obtained by using stochastic averaging method, from which a backward Kolmogorov equation governing the conditional reliability function and a Pontryagin equation governing the conditional mean of the first passage time are established. The conditional reliability function and the conditional mean of first passage time are obtained by solving these equations together with suitable initial condition and boundary conditions. Finally, an example of a 5-dimensional quasi-integrable generalized Hamiltonian system is worked out in detail and the solutions are confirmed by using a Monte Carlo simulation of the original system.

1. Introductions

The first passage problem is significant for structural reliability but it is very difficult to solve. The known exact solution is limited to the one-dimensional diffusion process. In the past decades, several numerical methods such as the generalized cell-mapping procedure [1] and Monte Carlo simulations [2] have been proposed to obtain the statistics of the first passage problem for higher-dimensional stochastic systems. At present, a powerful approximate technique for analyzing the first passage problem of higher-dimensional stochastic systems is the combination approach of the stochastic averaging method and the diffusion process theory of the first passage time, which has been applied by many authors, e.g., [3–8].

Many systems in science and engineering are of odd dimension, which can be modeled as stochastically excited and dissipated generalized Hamiltonian systems. Such systems may be classified into five groups based on the integrability and resonance of the associated generalized Hamiltonian systems. An m-dimensional quasi-integrable generalized Hamiltonian system is a generalized Hamiltonian system with M Casimir functions $C_1, \ldots, C_M$ and n ($n = (m - M)/2$) first integrals $H_{M+1}, \ldots, H_{M+n}$ in involution subjected to lightly linear and (or) nonlinear dampings and weakly stochastic excitations.

In the present paper, the equations governing a quasi-integrable generalized Hamiltonian system are reduced to a set of averaged Itô stochastic differential equations by using the stochastic averaging method. Then, the backward Kolmogorov equation governing the conditional reliability function and the Pontryagin equation governing the conditional mean of the first passage time are derived from the averaged equations. A 5-dimensional quasi-integrable generalized Hamiltonian system is taken as an example to illustrate the proposed procedure. The numerical results for the example are verified by using those from a Monte Carlo simulation.

2. Stochastic averaging

An m-dimensional dynamical system governed by

$$\dot{x}_i = J'_{ij}(\mathbf{x}) \frac{\partial H'}{\partial x_j}; \quad i, j = 1, \ldots, m \tag{1}$$

is called a generalized Hamiltonian system. In Eq. (1) $\mathbf{x} = [\mathbf{x}_1, \ldots, \mathbf{x}_m]^{\mathrm{T}}$ is a state vector; the dot denotes the derivative with respect to time t; $H' = H'(\mathbf{x})$ is a twice differentiable generalized Hamiltonian; $[J'_{ij}(\mathbf{x})]$ is an $m \times m$ anti-symmetric structural matrix, which satisfies the Jacobi identities [9] and, therefore, provides a generalized Poisson bracket

$$[F, G] = \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial F}{\partial x_i} J'_{ij}(\mathbf{x}) \frac{\partial G}{\partial x_j} \tag{2}$$

for two dynamical quantities $F(\mathbf{x})$ and $G(\mathbf{x})$ in phase space.

A function $F = F(\mathbf{x})$ is called a first integral of system (1) if $[F, H'] = 0$. A function $C = C(\mathbf{x})$ is called a Casimir function if $[C, G] = 0$, where $G = G(\mathbf{x})$ is any real-valued function. Obviously,

* Corresponding author. Tel.: +86 571 87953102.
E-mail address: wqzhu@yahoo.com (W.Q. Zhu).

0266-8920/$ – see front matter © 2009 Published by Elsevier Ltd
doi:10.1016/j.probengmech.2009.06.001

a Casimir function is a first integral of the system (usually generalized Hamiltonian systems). An m-dimensional integrable generalized Hamiltonian system has M Casimir functions $C_1, \dots, C_M$ and n $(n = (m-M)/2)$ independent first integrals $H_{M+1}, \dots, H_{M+n}$ in involution. The last term means that the generalized Poisson bracket of any two first integrals H_i and H_j vanishes, i.e., $[H_i, H_j] = 0$.

A quasi generalized Hamiltonian system is a generalized Hamiltonian system subjected to light dampings and weakly stochastic excitations governed by the following equations:

$$\dot{X}_i = J'_{ij}(\mathbf{X}) \frac{\partial H'}{\partial X_j} + \varepsilon \mathrm{d}'_{ij}(\mathbf{X}) \frac{\partial H'}{\partial X_j} + \varepsilon^{1/2} f_{is}(\mathbf{X}) W_s(t)$$
$$i, j = 1, \dots, m; \; s = 1, \dots, l \tag{3}$$

where $\mathbf{X} = [X_1, \dots, X_m]^{\mathrm{T}}$; $[J'_{ij}(\mathbf{X})]$ is an $m \times m$ anti-symmetric structural matrix; $H'(\mathbf{X})$ is a twice differentiable generalized Hamiltonian; $\varepsilon d'_{ij}(\mathbf{X})$ and $\varepsilon^{1/2} f_{is}(\mathbf{X})$ are the coefficients of quasi-linear dampings and the amplitudes of stochastic excitations, respectively; ε is a small positive parameter; $W_s(t)$ are Gaussian white noises in the sense of Stratonovich with correlation functions $E[W_s(t)W_z(t+\tau)] = 2D_{sz}\delta(\tau), s, z = 1, \dots, l$.

Eq. (3) can be modeled as Stratonovich stochastic differential equations and then converted into Itô stochastic differential equations by adding the Wong–Zakai correction terms $D_{sz} f_{js} f_{iz} / \dot{X}_j$. Splitting the Wong–Zakai correction terms into conservative part and dissipative part, and combining the two parts with $J'_{ij}(\mathbf{X})\partial H'/\partial X_j$ and $d'_{ij}(\mathbf{X})\partial H'/\partial X_j$, respectively, Eq. (3) is converted into the following Itô equations:

$$\dot{X}_i = \left[J_{ij}(\mathbf{X}) \frac{\partial H}{\partial X_j} + \varepsilon \mathrm{d}_{ij}(\mathbf{X}) \frac{\partial H}{\partial X_j} \right] \mathrm{d}t + \varepsilon^{1/2} \sigma_{is}(\mathbf{X}) \mathrm{d}B_s(t)$$
$$i, j = 1, \dots, m; \; s = 1, \dots, l \tag{4}$$

where $[J_{ij}(\mathbf{X})]$ is a modified structural matrix; H is a modified Hamiltonian; $\varepsilon d_{ij}(\mathbf{X})$ is the coefficients of modified quasi-linear dampings; $B_s(t)$ are the standard Wiener processes and $\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}} = 2\mathbf{f}\mathbf{D}\mathbf{f}^{\mathrm{T}}$.

Assume that the generalized Hamiltonian system governed by Eq. (4) with $\varepsilon = 0$ is integrable. Then, the Eq. (4) describes a quasi-integrable generalized Hamiltonian system. In principle, n $(n = (m-M)/2)$ pairs of action-angle variables I_k, Θ_k $(k = 1, \dots, n)$ can be introduced. Make the transformation

$$C_v = C_v(\mathbf{X}); \qquad I_{v_1} = I_{v_1}(\mathbf{X}); \qquad \Theta_{v_1} = \Theta_{v_1}(\mathbf{X})$$
$$v = 1, \dots, M; \; v_1 = 1 + M, \dots, n + M. \tag{5}$$

Then, the Itô equations for $C_1, \dots, C_M$, $I_{M+1}, \dots, I_{M+n}$, $\Theta_{M+1}, \dots, \Theta_{M+n}$ can be derived from Eq. (4) by using the Itô differential rule as follows:

$$\mathrm{d}C_v = \varepsilon U_v^{(1)} \mathrm{d}t + \varepsilon^{1/2} \sigma_{is} \frac{\partial C_v}{\partial X_i} \mathrm{d}B_s$$
$$\mathrm{d}I_{v_1} = \varepsilon U_{v_1}^{(1)} \mathrm{d}t + \varepsilon^{1/2} \sigma_{is} \frac{\partial I_{v_1}}{\partial X_i} \mathrm{d}B_s$$
$$\mathrm{d}\Theta_{v_1} = \left(\omega_{v_1} + \varepsilon U_{v_1}^{(2)}\right) \mathrm{d}t + \varepsilon^{1/2} \sigma_{is} \frac{\partial \Theta_{v_1}}{\partial X_i} \mathrm{d}B_s$$
$$v = 1, \dots, M; \; v_1 = M+1, \dots, M+n; \; s = 1, \dots, l \tag{6}$$

where

$$U_v^{(1)} = d_{ij} \frac{\partial C_v}{\partial X_i} \frac{\partial H}{\partial X_j} + \frac{1}{2} \sigma_{is}\sigma_{js} \frac{\partial^2 C_v}{\partial X_i \partial X_j}$$
$$U_{v_1}^{(1)} = d_{ij} \frac{\partial I_{v_1}}{\partial X_i} \frac{\partial H}{\partial X_j} + \frac{1}{2} \sigma_{is}\sigma_{js} \frac{\partial^2 I_{v_1}}{\partial X_i \partial X_j}$$
$$U_{v_1}^{(2)} = d_{ij} \frac{\partial \Theta_{v_1}}{\partial X_i} \frac{\partial H}{\partial X_j} + \frac{1}{2} \sigma_{is}\sigma_{js} \frac{\partial^2 \Theta_{v_1}}{\partial X_i \partial X_j} \tag{7}$$

where $\mathbf{X}$ on the right-hand side of Eq. (6) should be replaced by $C_1, \dots, C_M, I_{M+1}, \dots, I_{M+n}, \Theta_{M+1}, \dots, \Theta_{M+n}$ in terms of Eq. (5).

In the non-resonant case, $C_1, \dots, C_M, I_{M+1}, \dots, I_{M+n}$ are slowly varying processes while $\Theta_{M+1}, \dots, \Theta_{M+n}$ are rapidly varying processes. According to a theorem due to Khasiminskii [10] $C_1, \dots, C_M$, $I_{M+1}, \dots, I_{M+n}$ converge weakly to a $(M+n)$-dimensional diffusion process as $\varepsilon \to 0$ in a time interval $0 \le t \le T$, where $T \sim 0(\varepsilon^{-1})$. The Itô equations for an $(M+n)$-dimensional diffusion process are obtained by applying time averaging to Eq. (6) under the condition that the $C_1, \dots, C_M, I_{M+1}, \dots, I_{M+n}$ on the right-hand of Eq. (6) are kept constant. Since the phase flow of an integrable and non-resonant generalized Hamiltonian system is ergodic on the manifold of constant $I_{M+1}, \dots, I_{M+n}, C_1, \dots, C_M$, the time averaging can be replaced by phase space averaging with respect to $\Theta_{M+1}, \dots, \Theta_{M+n}$. Thus, the averaged Itô equations for $C_1, \dots, C_M, I_{M+1}, \dots, I_{M+n}$ are of the form

$$\mathrm{d}C_v = \varepsilon \bar{U}'_v(\mathbf{C}, \mathbf{I}) \mathrm{d}t + \varepsilon^{1/2} \bar{\sigma}'_{vs}(\mathbf{C}, \mathbf{I}) \mathrm{d}B_s$$
$$\mathrm{d}I_{v_1} = \varepsilon \bar{U}'_{v_1}(\mathbf{C}, \mathbf{I}) \mathrm{d}t + \varepsilon^{1/2} \bar{\sigma}'_{v_1 s}(\mathbf{C}, \mathbf{I}) \mathrm{d}B_s$$
$$v = 1, \dots, M; \; v_1 = M+1, \dots, M+n; \; s = 1, \dots, l \tag{8}$$

where

$$\bar{U}'_v = \left\langle d_{ij} \frac{\partial C_v}{\partial X_i} \frac{\partial H}{\partial X_j} + \frac{1}{2} \sigma_{is}\sigma_{js} \frac{\partial^2 C_v}{\partial X_i \partial X_j} \right\rangle_{\Theta}$$
$$\bar{U}'_{v_1} = \left\langle d_{ij} \frac{\partial I_{v_1}}{\partial X_i} \frac{\partial H}{\partial X_j} + \frac{1}{2} \sigma_{is}\sigma_{js} \frac{\partial^2 I_{v_1}}{\partial X_i \partial X_j} \right\rangle_{\Theta}$$
$$\bar{b}'_{v_2 v_3} = \bar{\sigma}'_{v_2 s}\bar{\sigma}'_{v_3 s} = \left\langle \sigma_{is}\sigma_{js} \frac{\partial C_{v_2}}{\partial X_i} \frac{\partial C_{v_3}}{\partial X_j} \right\rangle_{\Theta}$$
$$\bar{b}'_{v_2 v_1} = \bar{\sigma}'_{v_1 s}\bar{\sigma}'_{v_2 s} = \left\langle \sigma_{is}\sigma_{js} \frac{\partial C_{v_2}}{\partial X_i} \frac{\partial I_{v_1}}{\partial X_j} \right\rangle_{\Theta}$$
$$\bar{b}'_{v_4 v_5} = \bar{\sigma}'_{v_4 s}\bar{\sigma}'_{v_5 s} = \left\langle \sigma_{is}\sigma_{js} \frac{\partial I_{v_4}}{\partial X_i} \frac{\partial I_{v_5}}{\partial X_j} \right\rangle_{\Theta}$$
$$\langle \bullet \rangle_{\Theta} = \frac{1}{(2\pi)^n} \int_0^{2\pi} [\bullet] \, \mathrm{d}\Theta$$
$$v, v_2, v_3 = 1, \dots, M; \; v_1, v_4, v_5 = M+1, \dots, M+n. \tag{9}$$

Since $\mathbf{H} = \mathbf{H}(\mathbf{I})$, the equations for I_k in Eq. (8) can be replaced by those for H_k by using the Itô differential rule. The resultant averaged Itô equations for $\mathbf{C}$, $\mathbf{H}$ are of the following form:

$$\mathrm{d}C_v = \varepsilon \bar{U}_v(\mathbf{C}, \mathbf{H}) \mathrm{d}t + \varepsilon^{1/2} \bar{\sigma}_{vs}(\mathbf{C}, \mathbf{H}) \mathrm{d}B_s$$
$$\mathrm{d}H_{v_1} = \varepsilon \bar{U}_{v_1}(\mathbf{C}, \mathbf{H}) \mathrm{d}t + \varepsilon^{1/2} \bar{\sigma}_{v_1 s}(\mathbf{C}, \mathbf{H}) \mathrm{d}B_s$$
$$v = 1, \dots, M; \; v_1 = M+1, \dots, M+n; \; s = 1, \dots, l \tag{10}$$

where

$$\bar{U}_v = \bar{U}'_v; \qquad \bar{U}_{v_1} = \bar{U}'_{v_1} \frac{\partial H_{v_1}}{\partial I_{v_1}} + \frac{\bar{\sigma}'_{v_1 s}\bar{\sigma}'_{v_4 s}}{2} \frac{\partial^2 H_{v_1}}{\partial I_{v_1} \partial I_{v_4}}$$
$$\bar{b}_{v_2 v_3} = \bar{b}'_{v_2 v_3}; \qquad \bar{b}_{v_2 v_1} = \bar{b}'_{v_2 v_1} \frac{\partial H_{v_1}}{\partial I_{v_1}}$$
$$\bar{b}_{v_4 v_5} = \bar{b}'_{v_4 v_5} \frac{\partial H_{v_4}}{\partial I_{v_4}} \frac{\partial H_{v_5}}{\partial I_{v_5}}$$
$$v, v_2, v_3 = 1, \dots, M; \; v_1, v_4, v_5 = M+1, \dots, M+n \tag{11}$$

in which $\mathbf{I}$ are replaced by $\mathbf{H}$ in terms of $\mathbf{H} = \mathbf{H}(\mathbf{I})$.

3. Backward Kolmogorov equation

Generally, the generalized Hamiltonian H represents the total energy while the first integral H_{v1} and Casimir function C_v represent the energies of its sub-systems. Suppose that the averaged