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Probabilistic Engineering Mechanics 23 (2008) 393–407
PROBABILISTIC ENGINEERING MECHANICS
www.elsevier.com/locate/probengmech

# Critical review and latest developments of a class of simulation algorithms for strongly non-Gaussian random fields

Paolo Bocchini[a,1], George Deodatis[b,*]

[a] *DISTART, University of Bologna, Bologna, Italy*
[b] *Department of Civil Engineering and Engineering Mechanics, Columbia University, New York, NY, USA*

Received 17 April 2007; accepted 6 September 2007
Available online 21 February 2008

**Abstract**

A methodology is presented for simulation of strongly non-Gaussian random fields. It involves an iterative scheme that produces sample functions that match a prescribed non-Gaussian marginal distribution and a prescribed Spectral Density Function (SDF). The simulated field possesses all the properties of translation fields. The methodology also determines the SDF of an underlying Gaussian field according to translation field theory. This is the latest development in a class of simulation algorithms that are presented and critically reviewed. Several numerical examples are provided demonstrating the capabilities of the methodology, comparing it with three previous algorithms, and determining the limits of its applicability. Compared to earlier algorithms, the proposed methodology provides increased accuracy at a fraction of the computational cost.


*Keywords:* Non-Gaussian stochastic fields and processes; Spectral Representation Method; Translation fields and processes; Simulation

## 1. Introduction

The only universal method available to solve accurately any kind of stochastic mechanics problem is Monte Carlo Simulation (MCS). The only drawback of MCS is its computational cost, but lately the increased power of computers and the proliferation of clusters for parallel computing has strongly attenuated this issue (note also that the parallel implementation of MCS is generally straightforward).

One of the key parts in the implementation of the MCS methodology is the accurate and efficient generation of samples of the random processes and fields involved in the problem at hand. In stochastic mechanics problems, these random quantities can be, for example, random excitations and/or random material and geometric properties. To obtain accurate solutions to such problems, it is important that the generated sample functions of these processes and fields match the prescribed (target) probabilistic characteristics as closely as possible.

The most commonly used probability distribution to model these stochastic processes and fields is certainly the Gaussian. This choice is made mostly for convenience rather than for mathematical or physical reasons. For example, various material properties are bounded for physical reasons (e.g. the elastic modulus, the yield stress, the density, and several other properties cannot become negative), while excitations like wind pressure fluctuations and ocean wave heights are known to exhibit strong non-Gaussian characteristics. For these reasons, several methodologies have been proposed for simulating non-Gaussian stochastic processes and fields according to a prescribed Spectral Density Function (SDF) and a prescribed marginal Probability Distribution Function (PDF). A few representative methodologies along these lines are mentioned here in chronological order: Yamazaki and Shinozuka [1], Grigoriu [2,3], Gurley et al. [4], Popescu et al. [5], Gurley and Kareem [6], Deodatis and Micaletti [7], Puig et al. [8], Sakamoto and Ghanem [9,10], Graham et al. [11], Shi and Deodatis [12–14], Cope et al. [15], Li et al. [16].

This paper reviews critically three of the above methodologies [1,7,12] that have certain similarities and can be considered

* Corresponding author.
*E-mail addresses:* paolo.bocchini@unibo.it (P. Bocchini), deodatis@civil.columbia.edu (G. Deodatis).
[1] Currently Visiting Scholar at the Department of Civil Engineering and Engineering Mechanics, Columbia University, New York, NY, USA.

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doi:10.1016/j.probengmech.2007.09.001

to constitute a class of algorithms. They are all based on the concept of translation fields (Grigoriu [2]) and involve different iterative schemes that aim to generate non-Gaussian sample functions that reflect both the prescribed non-Gaussian PDF and SDF. An important part of these methodologies is to determine the SDF of an underlying Gaussian field according to translation field theory. The contribution of this work is to introduce the latest development in this class of algorithms, in the form of a methodology that provides increased accuracy and computational efficiency in matching the SDF and the PDF of the non-Gaussian field (compared to the three earlier versions).

In the remainder of this paper, only random fields will be considered for simplicity. The application to random processes is obviously trivial.

## 2. Critical review of previous research work

### 2.1. Spectral Representation Method for Gaussian fields

The Spectral Representation Method (SRM) for simulation of Gaussian stochastic processes and fields has been introduced by Shinozuka and Jan in 1972 [17]. Many further developments have been proposed over the years, which are reviewed and summarized in a series of papers by Shinozuka and Deodatis [18–20].

There are two alternative SRM-based algorithms in the literature (e.g. [18,21]). The one that will be used in this paper simulates the Gaussian stochastic field $g(x)$ by the following series as $N \to \infty$:

$$g(x) = 2 \sum_{n=0}^{N-1} \sqrt{S_{gg}(\kappa_n)\, \Delta\kappa} \cos(\kappa_n x + \phi_n) \tag{1}$$

where

$$\Delta\kappa = \frac{\kappa_u}{N} \tag{2}$$

$$\kappa_n = n\, \Delta\kappa, \quad n = 0, 1, \ldots, N-1 \tag{3}$$

and $\kappa_u$ is the upper cutoff wave number, beyond which the SDF $S_{gg}(\kappa)$ can be assumed to be zero for mathematical or physical reasons. The $\phi_n$'s are independent random phase angles, uniformly distributed in the interval $[0, 2\pi]$. The simulated stochastic field is periodic with period:

$$L = \frac{2\pi}{\Delta\kappa}. \tag{4}$$

Generated sample functions are Gaussian only asymptotically due to the central limit theorem. A value of $N = 256$ is used in this paper with reasonably accurate results regarding Gaussianity, following an extensive numerical investigation. Generated sample functions have also a strong ergodic property in the mean and autocorrelation when the length of the sample is a multiple of the period $L$ or tends to infinity [18,21]. It is worthwhile noting that this ergodic property is lost when the Gaussian sample function is mapped to a non-Gaussian one according to the classic translation process transformation.

### 2.2. Spectral Representation Method for non-Gaussian fields: Yamazaki and Shinozuka (1988)

In 1988 Yamazaki and Shinozuka [1] proposed an SRM-based iterative methodology to simulate a non-Gaussian stochastic field $f(x)$ according to a target non-Gaussian SDF, $S_{ff}^T(\kappa)$, and a target non-Gaussian marginal CDF, $\mathcal{F}_f$, with zero mean and variance $\sigma_f^2$ compatible with that of the target SDF. Their methodology is based on Grigoriu's translation process theory [22,2].

At the first iterative step, the unknown SDF of the underlying Gaussian field $g(x)$, $S_{gg}(\kappa)$, is set equal to $S_{ff}^T(\kappa)$. Then, a Gaussian sample function $g(x)$ is generated by means of the SRM. The classic translation process transformation [22,2] is then used to map the homogeneous Gaussian sample into a homogeneous non-Gaussian one with the prescribed marginal PDF:

$$f(x) = \mathcal{F}_f^{-1}\left\{\mathcal{F}_g\left[g(x)\right]\right\} \tag{5}$$

where $\mathcal{F}_f^{-1}$ is the inverse target non-Gaussian Cumulative Distribution Function (CDF) and $\mathcal{F}_g$ is the Gaussian CDF with zero mean and variance $\sigma_g^2$ equal to $\sigma_f^2$. Although sample function $f(x)$ reflects the prescribed non-Gaussian marginal CDF, $\mathcal{F}_f$, its SDF is not matching, in general, the prescribed SDF, $S_{ff}^T(\kappa)$, because of the nonlinearity of the transformation in Eq. (5).

The basic idea of the Yamazaki and Shinozuka algorithm is to update iteratively the SDF of the underlying Gaussian field until the SDF of the non-Gaussian sample function converges to the target. This is expressed as:

$$S_{gg}^{(j+1)}(\kappa) = \frac{S_{gg}^{(j)}(\kappa)}{S_{ff}^{(j)}(\kappa)} S_{ff}^T(\kappa) \tag{6}$$

where $S_{gg}^{(j+1)}(\kappa)$ and $S_{gg}^{(j)}(\kappa)$ denote the SDF's of the underlying Gaussian field at the $(j+1)$th and $j$th iterations, respectively, and $S_{ff}^{(j)}(\kappa)$ is the SDF of the non-Gaussian sample function at the $j$th iteration computed from:

$$S_{ff}^{(j)}(\kappa) = \frac{1}{2\pi L} \left| \int_0^L f^{(j)}(x) \exp(-\mathrm{i}\kappa x)\, \mathrm{d}x \right|^2. \tag{7}$$

Over the years since its introduction in 1988, it has become clear that the Yamazaki and Shinozuka algorithm cannot match accurately the prescribed non-Gaussian marginal CDF when it deviates significantly from the Gaussian. Deodatis and Micaletti [7] have identified and explained in detail the theoretical reasons for this problem that are briefly summarized here.

The first reason is that the underlying "Gaussian" field diverges from Gaussianity as the iterations proceed. This is due to the fact that the updating formula in Eq. (6) makes the SDF $S_{gg}(\kappa)$ of the underlying Gaussian field a function of all the $\phi_n$'s in Eq. (1) (since $S_{ff}^{(j)}(\kappa)$ is a function of all the $\phi_n$'s as can be seen from Eq. (7)). Once $S_{gg}(\kappa)$ becomes a function of all the $\phi_n$'s, each term in the summation in Eq. (1) becomes a