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An island particle algorithm for rare event analysis

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ABSTRACT

Estimating rare event probability with accuracy is of great interest for safety and reliability applications. In this paper, we focus on rare events which can be modeled by a threshold exceedance of a deterministic input–output function with random inputs. Some parameters of this function or density parameters of input random variables may be fixed by an experimenter for simplicity reasons. From a risk analysis point of view, it is not only interesting to evaluate the probability of a critical event but it is also important to determine the impact of such tuning of parameters on the realization of a critical event, because a bad estimation of these parameters can strongly modify rare event probability estimations. In the present paper, we present an example of island particle algorithm referred to as sequential Monte Carlo square (SMC^2). This algorithm gives an estimate of the law of random phenomena that leads to critical events. The principles of this statistical technique are described throughout this article and its results are analysed on different realistic aerospace test cases.

1. Introduction

Safety and reliability have received increasing attention in various fields as aerospace [56], aeronautics [62], genetics [65], finance [63,32], insurance [2], geological phenomena [2,39], informatic networks [61] or nuclear domain [69]. The recent security rules or constraints in all those areas are such that the failure of a complex system must be a rare event. It is thus essential to be able to compute small probabilities. The most popular technique to estimate a probability p is the Crude Monte Carlo (CMC) which consists in approximating a probability by an empirical mean of N independent and i.i.d. samples. CMC is simple to implement and provides an unbiased and consistent estimate of the probability of interest p. The relative error (which is the ratio between the standard deviation of an estimator and its mean) measures the variability of the estimation with respect to (w.r.t.) the quantity to estimate. It is an indicator for the quality of the estimation. The relative error of the CMC is of order $1/\sqrt{Np}$, so that in order to estimate a low probability $p = 10^{-k}$ ($k \in \mathbb{N}$) with a relative error of 10%, a very large sample $N = 10^{k+2}$ is needed. In the rare event context, the CMC is thus inefficient and a lot of fast simulation and variance reduction techniques have been proposed to estimate small probabilities since 1950s. Indeed, importance sampling [26,60,63,43] is one of the most famous methods and is the basics of a lot of other methods as the Esscher transformation for compound Poisson processes [2,31,9], the transformations for random Gaussian vectors [38] or sequential Monte Carlo (SMC) methods [65,35,25]. Splitting (also called subset simulation) is an example of SMC methods and has been considerably studied and improved over the past decades, see the references [40,66,28,53,33,29,4,15,45,46,16,13,3,34]. Different techniques as Markov chain Monte Carlo (MCMC), First and Second Order Reliability Methods (FORM/SORM) [8,52,77], or extreme value theory [27,47,48] are also well-known simulation algorithms to estimate very small probability of failure. Their principles and advantages/drawbacks have also been deeply studied, see for example [58].

In this paper, we focus on rare events which can be modeled by a threshold exceedence of a deterministic output function. This "input–output" function can be seen as a "black-box" with random inputs. Some parameters, denoted by a vector Θ, in black-box functions are implicitly set, such as parameters of the model or of the input parametric model density, and their value influences the rare event probability estimation. These hypotheses are often assumed for simplification and computational reasons. From a risk analysis point of view, it is interesting to determine the variability of the output of a code w.r.t. the uncertainty on these input parameters Θ or w.r.t. one particular parameter, and to quantify the impact of such tuning in the realization of a critical event. Of course, different values of Θ can strongly modify rare event probability estimation and sometimes miss a risk situation. The issue of concern in safety would be to underestimate a risk

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because of a bad tuning of model parameters Θ. That is why in this paper we propose to estimate the law of the parameters conditionally on a critical event, and to present different algorithms for this purpose.

Some studies on close topics have been proposed recently, mostly based on sensitivity analysis methods. Monte Carlo filtering [68] consists in determining the differences between a "safe" sample and a "faulty" sample *via* standard statistical tests. The reliability index resulting from FORM/SORM [50] can also be used to analyze the influence of input parameters on the failure probability. Stratified sampling [59] and importance sampling [56] have been adapted with the same purpose. See [6] and the references wherein for an in-depth presentation of parameter estimation methods. The analysis of parameter sensitivity is challenging in numerous domains such as molecular phylogenetic sequence analysis [75], systems biology or postgenomic era [18], and basin hydrology design [7].

The methods proposed in this paper are rather different. The objective of sensitivity analysis is to determine how different values of parameters Θ will impact (increase or decrease) the rare event probability under a given set of assumptions. Such approach has been proposed recently in [44] with subset sampling or [56] with importance sampling and Sobol indices. It is not the case here since one focuses more precisely on the settings of the parameters Θ that leads to high rare event probability values but have also a high likelihood. This trade-off is very important in safety and reliability. For instance, improbable settings of parameters Θ that maximizes the rare event probability are not of interest. In the same way, likely settings of parameters Θ that decreases the rare event probability are not useful for analysing and bounding the risk. For that purpose, the proposed approach is to estimate the distribution of these input parameters Θ conditionally on the rare event of interest and to compare it to the initial distribution of Θ.

We will see in Section 3.1 why this distribution enables us to find a good balance between maximizing the rare event probability and keeping a high likelihood for the parameters Θ. We present two methods to estimate this kind of targeted laws derived from the PMCMC and the SMC^2 algorithms, respectively introduced by [1] and [20] to do filtering on hidden Markov models. They are respectively noisy versions of MCMC and SMC algorithms where intractable quantities are replaced by unbiased estimators produced by an auxiliary SMC algorithm.

MCMC consists in constructing a Markov chain whose transition kernels leave the target density of interest invariant. The most common examples are the Metropolis–Hastings (MH) algorithm and the Gibbs sampler. In their implementation, they both require the evaluation of the probability of the rare event for different proposals of parameters. As it cannot be computed in our context, it must be estimated by a method designed for rare event estimation, and the algorithm becomes the PMCMC. In this paper, we have chosen to implement splitting in order to estimate rare event probabilities.

SMC algorithms consist in running a batch of particles defined on the parameter space, and to select and update these parameters through recursive selection and mutation steps, in order to sequentially approximate the targeted laws of interest. In this special framework, for each value of parameter in the sample we need to compute the probability of the rare event given this parameter, which can be estimated by another SMC algorithm (such as the splitting method). We then have two embedded SMC algorithms that justify the terminology SMC^2 algorithm. It is also an example of island particle algorithm introduced by [73].

To our knowledge, the development of these methods for rare event case and safety application has not been proposed yet. We prove the validity of these algorithms in Sections 3.1 and 3.2.2. We illustrate the convergence of these two algorithms on a test case. The main advantage of the SMC^2 algorithm is to bypass the long time convergence issue of the PMCMC algorithm using interacting island particle model.

In the paper, we firstly raise the fundamental question : how to estimate the law of Θ conditionally on a rare event? Two algorithms, PMCMC and SMC^2, are proposed to solve this issue. Their principle and algorithmic implementation are described, and the performance of these algorithms are discussed on a test case. We also apply the SMC^2 algorithm to two realistic simulations in the aerospace domain : the estimation of fallout position of a stage of a launch vehicle and the estimation of a space collision between a satellite and a space debris.

2. General problem

In this section, we underline the challenge of estimating the conditional distribution of the parameters of a model given a rare event in reliability, and we set notations.

Let us consider a d-dimensional random variable X defined on a measurable space $(\mathbb{X}, \chi)$, ϕ a continuous positive scalar function $\phi : \mathbb{R}^d \to \mathbb{R}$ and $S \in \mathbb{R}^+$ a given critical threshold. The function ϕ is static, i.e. does not depend on time, and represents for instance an input–output model. We assume that the output $Y = \phi(X)$ is also a positive random variable (typically a distance). This kind of model is notably used in various applications [41,42,5,36,76]. The quantity of interest on the output Y is the probability of exceedance

$$\mathbb{P}(Y > S) = \mathbb{P}(\phi(X) > S).$$

When the event $\{Y > S\}$ is rare relatively to the available simulation budget (which is often the case in safety and reliability issues), different algorithms described in [71,67,78,8,10,12,13] have been proposed to estimate accurately its probability.

In the present paper, one focuses on the case where the law of X is uncertain and depends upon unknown parameters. We assume that X is distributed according to a well known parametric model and its parameters, denoted by a random vector Θ, have a probability density ν. We also suppose that Θ has a density f_Θ w.r.t. a dominating measure of reference λ, that is

$$\nu(d\theta) = f_\Theta(\theta)\lambda(d\theta).$$

For instance, in the applications considered in this paper, X is a random vector with a multivariate normal distribution, and Θ may describe the mean or the covariance matrix of X. It corresponds to realistic applications where it is not always possible to evaluate accurately the density of input parameters. This formalism enables thus to consider a large range of input probability density function.

The probability of interest $\mathbb{P}(Y > S)$ depends of course on Θ and thus on the distribution ν. In safety applications, it is important to estimate a superior bound of the rare event probability $\mathbb{P}(Y > S)$ taking also into account the prior on Θ. The prior on Θ is important since unrealistic bad tuning values of Θ which lead to high probabilities $\mathbb{P}(Y > S)$ are not relevant. The idea of this paper is thus to determine the distribution of Θ conditionally on the fact that Y exceeds the threshold S. This distribution, denoted by π, will be referred to in the sequel as the *target law*. A possible approach to analyse the influence of Θ on the rare event probability is to compare ν and π. If ν and π are similar, it means that ν is a distribution that leads to high realistic probabilities $\mathbb{P}(Y > S)$. The worst situation in term of safety is thus already considered in the model and thus any change of ν tends to decrease the rare event probability $\mathbb{P}(Y > S)$. If ν and π are rather different, the rare event probability could strongly increase and it is a potentially dangerous situation for the system safety. In this paper, we propose to measure the difference between ν and π with the Kullback–Leibler divergence even if other metrics on the probability space could have been chosen instead.