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A flexible Weibull extension

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Abstract

We propose a new two-parameter ageing distribution which is a generalization of the Weibull and study its properties. It has a simple failure rate (hazard rate) function. With appropriate choice of parameter values, it is able to model various ageing classes of life distributions including IFR, IFRA and modified bathtub (MBT). The ranges of the two parameters are clearly demarcated to separate these classes. It thus provides an alternative to many existing life distributions. Details of parameter estimation are provided through a Weibull-type probability plot and maximum likelihood. We also derive explicit formulas for the turning points of the failure rate function in terms of its parameters. This, combined with the parameter estimation procedures, will allow empirical estimation of the turning points for real data sets, which provides useful information for reliability policies.

Keywords: Reliability; IFR; IFRA; Bathtub shape; Weibull-type probability plot; Maximum likelihood estimates

1. Introduction

Motivated by engineering applications, Weibull [1] suggested a distribution that has proved to be of seminal importance in reliability. The survival function is given by the equation

$$\bar{F}(t) = \exp(-(t/\beta)^{\alpha}), \quad t > 0, \tag{1}$$

with parameters $\alpha, \beta > 0$. The corresponding failure rate function is

$$h(t) = \frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1}. \tag{2}$$

The Weibull distribution has been widely used to model the failures of many materials, and in numerous other applications. A very small sample of the vast literature includes applications to the yield strength and fatigue life of steel [1], fracture strength of glass [2], pitting corrosion in pipes [3], adhesive wear in metals [4], and failure of carbon fibre composites [5], coatings [6], brittle materials [7], composite materials [8], and concrete components [9]. Recently, it has been applied in mixture models, particularly for automobile warranty data [10–12].

A recent paper by Murthy et al. [13] discusses additional applications, and gives a methodological review of the 'Weibull area'. It also suggests further study of various Weibull-type distributions, their properties, related plots, and model selection. In this paper we introduce yet another member of the Weibull family, which we call the 'flexible Weibull distribution'. We will first define it mathematically, then examine its properties, and finally details of its application.

We see from (2) that the Weibull distribution has a monotonic failure rate function, although this may be increasing or decreasing. In more complex systems, such as electronic ones, the failure rate is often non-monotonic. This usually takes the form of increased failure rate early ('wear-in') and late ('wear-out') in the component lifetime. This is usually termed a 'bathtub-shaped' failure rate, or if the limit of the failure rate at time zero is zero, a 'modified bathtub' (MBT) (or 'roller coaster')-shaped failure rate. Given the utility of the bathtub-shaped failure rate functions in reliability engineering, many of the variations on the Weibull distribution have been motivated by the desire to produce a bathtub-shaped failure rate function.

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Gurvich et al. [14] introduced a class of distributions characterized by the cumulative distribution function

$$F(t) = 1 - \exp\{-aG(t)\}, \quad t > 0, \tag{3}$$

with parameter $a > 0$, where $G(t)$ is a monotonically increasing function of t. Equivalently, the survival function is $\bar{F}(t) = \exp\{-aG(t)\}$.

Nadarajah and Kotz [15] have since shown that several existing life distributions, such as the modified Weibull extension $F(t) = 1 - \exp(-\lambda\alpha[(t/\alpha)^\beta - 1])$, see [16], may be expressed in the form (3). Clearly, (3) is a generalized Weibull distribution. Xie et al. [17] reviewed several families of extended Weibull distributions, while a comprehensive taxonomy of Weibull models can be found in [18].

In this paper, we propose a new life distribution of a similar form to (3), with the difference that $G(t)$ is not a monotonic function of t. We will completely characterize the failure rate function, and consider parameter estimation. The new distribution is shown to be quite flexible, being able to model both IFR and IFRA ageing classes. Also, it can yield a MBT-shaped failure rate distribution, and in particular, allows considerable flexibility in modelling the 'pre-useful' (i.e., infancy) period.

2. The new flexible Weibull distribution

We will define our model in terms of the survival function. Let T be a lifetime random variable, and

$$\bar{F}(t) = \exp(-e^{\alpha t - \beta/t}), \quad t > 0, \tag{4}$$

with parameters $\alpha, \beta > 0$. Clearly, this has the form of distribution (3) with $a = 1$ and

$$G(t) = e^{\alpha t - \beta/t}. \tag{5}$$

However, function (5) is not monotonic in t and so the distribution under consideration differs from that in (3) considered by Gurvich et al. [14]. Note that when $\beta = 0$, if we set $\alpha = \log(\lambda)$, distribution (4) becomes exponential and thus the proposed life distribution may be regarded as a generalization of the Weibull.

The density function corresponding to (4) is

$$f(t) = (\alpha + \beta/t^2)\exp(\alpha t - \beta/t)\exp(-e^{\alpha t - \beta/t}).$$

Though formulas for the mean and the variance are difficult to obtain explicitly, the quantiles are easy to evaluate. Let t_p be the pth quantile of T. By considering the log–log transformation of $p = \bar{F}(t)$, we have $\log(-\log p) = (\alpha t^2 - \beta)/t$ and so t_p is a solution of the quadratic equation in t

$$\alpha t^2 - \log(-\log p)\, t - \beta = 0.$$

Since the solutions have to be non-negative, the only one is

$$t_p = \frac{1}{2\alpha}\left(\log(-\log p) + \sqrt{\{\log(-\log p)\}^2 + 4\alpha\beta}\right).$$

The failure rate function has a reasonably simple form

$$h(t) = \frac{f(t)}{\bar{F}(t)} = (\alpha + \beta/t^2)\exp(\alpha t - \beta/t). \tag{6}$$

The shape of the density and failure rate function are illustrated for selected values of α and β in Figs. 1 and 2. For reference, the solid curve is the same in both figures. We see that as β decreases, the failure rate function becomes more 'bathtub-like'. While, as α increases, the 'bathtub' becomes 'shallower'. In particular, we see that the model has a great deal of flexibility in the existence and weight of the failure mode corresponding to wear-in. Various properties of $f(t)$ and $h(t)$ that follow from their definitions and/or can be seen in Figs. 1 and 2 are discussed in detail in the next section.

3. Ageing behaviour

We note that $\lim_{t\to\infty} h(t) = \infty$, thus the failure rate function is ultimately increasing. Also, with the notation $s = 1/t$ for computational convenience,

$$\begin{aligned}\lim_{t\to 0} h(t) &= \lim_{s\to\infty} (\alpha + \beta s^2)\exp(-\beta s + \alpha/s)\\ &= \lim_{s\to\infty} 2\beta s(\beta + \alpha/s^2)^{-1}\exp(-\beta s + \alpha/s)\\ &= \lim_{s\to\infty} 2\beta(\beta + 2\alpha/s^2 - 2\alpha/s^3 + \alpha^2/s^4)^{-1}\\ &\quad \times \exp(-\beta s + \alpha/s)\\ &= 0,\end{aligned}$$

and thus a pure bath-tub form is impossible.

The rest of this section is subdivided into three subsections in which we consider more subtle properties of the failure rate function $h(t)$. First we determine the values of α and β for which $h(t)$ is increasing, that is, belongs to the IFR class, and then we determine the values of α and β for which $h(t)$ belongs to the IFRA class. Interestingly, we shall show that there are values of α and β for which $h(t)$ is IFRA but not IFR. Finally, we will consider in detail the case when $h(t)$ is not IFR, and show that $h(t)$ takes on the MBT shape. Such failure rate functions are particularly useful models in many practical situations (cf., e.g., [19]).

3.1. IFR

We need to determine the parameter values for which the failure rate function $h(t)$ is an increasing function, that is, belongs to the IFR class. Naturally, we solve the problem by specifying those α and β for which $h'(t)$ is strictly positive for all $t \geqslant 0$.

Consider the derivative of the failure rate function $h(t)$. It follows from (6) that

$$\begin{aligned}h'(t) &= \frac{-2\beta}{t^3}e^{\alpha t - \beta/t} + \frac{\alpha t^2 + \beta}{t^2}(\alpha + \beta/t^2)e^{\alpha t - \beta/t}\\ &= \frac{(\alpha t^2 + \beta)^2 - 2\beta t}{t^4}e^{\alpha t - \beta/t}.\end{aligned}$$