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# A Haar wavelet method for angularly discretising the Boltzmann transport equation

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ABSTRACT

A novel, hierarchical Haar wavelet basis is introduced and used to discretise the angular dimension of the Boltzmann transport equation. This is used in conjunction with a finite element subgrid scale method. This combination is then validated using two steady-state radiation transport problems, namely a 2D dogleg-duct shielding problem and the 2D C5MOX OECD/NEA benchmark. It is shown that the scheme has many similarities to a traditional equal weighted discrete ordinates ($S_n$) angular discretisation, but the strong motivation for our hierarchical Haar wavelet method is the potential for adapting in angle in a simple fashion through elimination of redundant wavelets. Initial investigations of this adaptive approach are presented for a shielding and criticality eigenvalue example. It is shown that a 60% reduction in the number of angles needed on most spatial nodes - and rising up to 90% on nodes located in high streaming areas - can be attained without adversely affecting the accuracy of the solution.

## 1. Introduction

Modern wavelet research was pioneered by the likes of Morlet, Grossman, and Daubechies (Grossmann and Morlet, 1984; Daubechies, 1988). The research efforts of the former two in signal processing led to the creation of a wavelet transform method that allowed the representation of information localised simultaneously in time and frequency, a feat otherwise not possible using Fourier Transforms (Grossmann and Morlet, 1984). The wavelet transform method was also unique in that the wavelet used as the analysing functions were not limited to a particular type. This spurred the proposal of several wavelet types (or families) differing in form from constant functions to polynomials and exponentials (Kronland-Martinet et al., 1987; Schroder and Sweldens, 1995). Some examples of wavelet families include the Legendre, Morlet, Gaussian, Shannon, Haar, and Daubechies wavelets (Aboufadel and Schlicker, 1999).

While the proposed wavelet families differed individually, they all shared the property of being zero everywhere except on a small interval. Such functions with zero values everywhere outside their closed and bounded intervals were said to be *compactly* supported (Aboufadel and Schlicker, 1999). Daubechies' work would later produce a groundbreaking paper on compactly supported wavelets, proving that it was possible to develop wavelets with desired properties tailored to specific applications (Daubechies, 1988). The research work which ensued led to the development of the multiresolution analysis (MRA) which enabled a hierarchical wavelet reconstruction of a signal in layers of increasing resolution. The second-generation wavelet transform was later developed to address some of the deficiencies of traditional wavelets, allowing them to be used in representing functions on arbitrary domains (Schroder and Sweldens, 1995).

Before long, wavelets became a very useful tool in a number of fields such as image/data compression and computer graphics applications, and over time, expanded into a number of numerical analysis fields (Aboufadel and Schlicker, 1999; Buchan et al., 2005; Li and Chen, 2014). In solving ordinary differential equations and partial differential equations, wavelets have been applied for the Navier-Stokes equation (Schneider and Farge, 2000; Wei et al., 1998; Zhou and He, 2005) and other parabolic (Heydari et al., 2014; Ho and Yang, 2001; Chiavassa et al., 2002) and hyperbolic equations (Alves et al., 2002; Hong and Kennett, 2002; Massel, 2001; Reckinger et al., 2014). More recently, wavelets have been applied in solving the Boltzmann Transport Equation (BTE) (Buchan et al., 2005, 2011; Cao et al., 2008; Zheng et al., 2009a, 2009b; Adam et al., 2016). Schroder and Sweldens (1995) spearheaded the construction of wavelets on the sphere to represent spherical functions. Some other works showing the different applications of wavelets on the sphere include Freeden and Windheuser

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(1997), Demanet et al. (2002), Antoine and Vandergheynst (1999), and Buchan et al. (2008).

This article presents a new approach for representing the direction of neutron particle travel through a Haar wavelet discretisation of the angular dimension of the BTE. Related works include that of Buchan et al. who applied linear and quadratic octahedral wavelets (Buchan et al., 2005) as well as self-adaptive spherical wavelets (Buchan et al., 2008) on the sphere to represent the angular flux of the BTE. Cao et al. used a Daubechies' double wavelets expansion in solving the angular domain of the neutron transport equation (Cao et al., 2008). Recently, a two-dimensional Haar wavelet collocation method has been used by Patra and Ray to solve the neutron transport (Patra and Ray, 2014a) and point kinetics equation (Patra and Ray, 2014b). However, this is the first time the angular variable of the neutron flux has been discretised through a dual Haar wavelet expansion on the sphere combined with a subgrid scale (SGS) finite element method (FEM) that enables an efficient solution method.

The dual Haar wavelet discretisation can be shown to generate identical discretisations to the $S_n$ method when particular weights and directions are used. However a strong motivation for pursuing this Haar wavelet approach is that its inherent properties of using compactly supported functions within a hierarchical expansion scheme allow the incorporation of angular adaptivity into the discretisation. Compactly supported functions locally resolve the problem meaning that only those that contribute most to the solution need to be retained. Redundant wavelet functions that contribute little to the solution can be eliminated from the calculation and this can be achieved in a consistent manner through the use of hierarchical expansions, as shown in (Buchan et al., 2008). As such, resolution can be focused on regions of angle where it is needed most using fewer functions than traditional approaches to enhance computational efficiency. This Haar approach also has advantages over other similar wavelet schemes, e.g. (Buchan et al., 2008), as it leads to more sparse discretised matrix systems that can be utilized to improve solving times. It also bridges the gap between $S_n$ and adaptivity in angle, although other important works in this area include that of Adams and Larsen (2002), Jessee et al. (1998), Ragusa and Wang (2010).

The aim of this article is to lay the foundations by developing the Haar wavelet angular discretisation for the resolution of the BTE. It will show that the Haar wavelet method can produce solutions that are identical to certain $S_n$ quadrature schemes. However it will also show these quadrature rules are not necessarily optimal in comparison to other $S_n$ methods. The most important component for the Haar wavelets' use in neutron transport is therefore their potential to adapt in angle, which may substantially reduce problem size and decrease computational solving times. For certain problem types angular adaptivity could exceed standard $S_n$ methods when constant and high resolution quadrature sets are used. To lay the foundations this paper focuses on two developments in addition to the Haar wavelet discretisation. Firstly it will develop the angular Haar wavelet method within a spatial discretisation scheme, suitable for the BTE, that can be solved efficiently without a sweep based solver. This is vital as the Haar wavelets will introduce coupling of angular moments through the streaming operator and sweep-based $S_n$ like solvers can no longer be employed. Secondly we will demonstrate the potential of adaptivity by filtering redundant wavelets from their solution. This aims to highlight the powers of adaptivity and indicate the reduction in problem size that may be achieved. A second article will follow the developments here showing how the model and solver can be adapted to enable self-adaptivity that produces highly efficient solutions in angular resolution.

The sections of this article are set out as follows. In section 2 the BTE is introduced and the discretisation of the space and angle dimensions are given. In section 3, the MRA is introduced and the general wavelet MRA is applied to the Haar wavelet family in capturing the angular dependence of the BTE. In section 4, two numerical examples are presented. These are specifically chosen to demonstrate the capability of the Haar approach over a range of radiation transport conditions and to illustrate the potential gains from using adaptivity with Haar wavelets in this framework. Finally, section 5 completes the paper with a conclusion on the findings and proposed future works.

## 2. Discretisation of the Boltzmann transport equation

The following sections describe the BTE and provide an overview of the space-angle discretisation methods used in this article.

### 2.1. The Boltzmann transport equation

The BTE governs the conservation of neutral particle transport within its surrounding medium. For fixed source problems, the first order time-independent multi-group equations are given as (Buchan et al., 2015; Goffin et al., 2013)

$$\hat{\mathbf{\Omega}}\cdot\nabla\psi_g(\mathbf{r},\hat{\mathbf{\Omega}}) + \Sigma_{t,g}(\mathbf{r})\psi_g(\mathbf{r},\hat{\mathbf{\Omega}}) = \int_{4\pi}\sum_{g'=1}^{G}\Sigma_{s,g'\to g}(\mathbf{r},\hat{\mathbf{\Omega}}'\to\hat{\mathbf{\Omega}})\psi_{g'}(\mathbf{r},\hat{\mathbf{\Omega}}')\,\mathrm{d}\hat{\Omega}' + S_{g'}(\mathbf{r},\hat{\mathbf{\Omega}}),\quad \forall\ g\in\{1,2,3\ldots,G\}, \tag{1}$$

where G energy groups are represented and the subscript g denotes each energy group. The angular flux, $\psi_g(\mathbf{r},\hat{\mathbf{\Omega}})$, is defined over a five dimensional solution space; three of which are in the spatial direction $\mathbf{r}$, and two of which are in the direction of travel or angular domain $\hat{\mathbf{\Omega}}$. The cross section, $\Sigma_t$, defines the probability that the particles are removed through both absorption and scattering, (i.e. $\Sigma_t = \Sigma_a + \Sigma_s$) with the source term designated as $S_g$. For criticality problems, the eigenvalue form of the BTE reads as,

$$\hat{\mathbf{\Omega}}\cdot\nabla\psi_g(\mathbf{r},\hat{\mathbf{\Omega}}) + \Sigma_{t,g}(\mathbf{r})\psi_g(\mathbf{r},\hat{\mathbf{\Omega}}) = \int_{4\pi}\sum_{g'=1}^{G}\Sigma_{s,g'\to g}(\mathbf{r},\hat{\mathbf{\Omega}}',\to\hat{\mathbf{\Omega}})\psi_{g'}(\mathbf{r},\hat{\mathbf{\Omega}}')\,\mathrm{d}\hat{\Omega}' + \lambda\frac{\chi_g}{4\pi}\sum_{g'=1}^{G}\nu\Sigma_{f,g'}(\mathbf{r})\int_{4\pi}\psi_{g'}(\mathbf{r},\hat{\mathbf{\Omega}}')\,\mathrm{d}\hat{\Omega}',\quad \forall\ g\in\{1,2,3\ldots,G\}, \tag{2}$$

where the fission energy spectrum is denoted as $\chi_g$, $\nu_g$ is the average number of neutrons emitted per fission and $\Sigma_{f,g}$ the fission cross-section. We consider both the angular flux $\psi_g$ and eigenvalue $\lambda = 1/k_{\mathrm{eff}}$. In the following sections it is sufficient to consider just the fixed source mono-energetic equations.

Both vacuum and reflective boundary conditions will be considered in this article which are defined as,

$$\psi(\mathbf{r},\hat{\mathbf{\Omega}}) = 0,\quad \text{where}\quad \hat{\mathbf{\Omega}}\cdot\mathbf{n} < 0. \tag{3}$$

and,

$$\hat{\mathbf{\Omega}}\cdot\mathbf{n} = -\hat{\mathbf{\Omega}}'\cdot\mathbf{n}\quad \text{and}\quad (\hat{\mathbf{\Omega}}'\times\hat{\mathbf{\Omega}})\cdot\mathbf{n} = 0, \tag{4}$$

respectively. The term $\mathbf{n}$ denotes the outward facing normal to the boundary and $\hat{\mathbf{\Omega}}'$ is the specular reflected angle to $\hat{\mathbf{\Omega}}$ with respect to $\mathbf{n}$. Satisfying these conditions using the wavelet discretisation is discussed in section 3.

The angular dimension $\hat{\mathbf{\Omega}}$ is conventionally represented on the