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# Galerkin and Generalized Least Squares finite element: A comparative study for multi-group diffusion solvers

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ABSTRACT

In this paper, the solution of multi-group neutron/adjoint equation using Finite Element Method (FEM) for hexagonal and rectangular reactor cores is reported. The spatial discretization of the neutron diffusion equation is performed based on two different Finite Element Methods (FEMs) using unstructured triangular elements generated by Gambit software. Calculations are performed using Galerkin and Generalized Least Squares FEMs; based on which results are compared. Using the power iteration method for the neutron and adjoint calculations, the neutron and adjoint flux distributions with the corresponding eigenvalues are obtained. The results are then validated against the valid results for the IAEA-2D andBIBLIS-2D benchmark problems. The results of GFEM-2D (developed based on Galerkin FEM) and GELES-2D (developed based on Generalized Least Squares FEM) computer codes are also compared with results obtained from DONJON4 computer code. To investigate the validation of developed computer codes for the calculation with more than two energy groups, the calculations are performed for a benchmark problem with seven energy groups. To investigate the dependency of the results to the number of elements, a sensitivity analysis of the calculations to the number of elements is performed.



## 1. Introduction

Numerical solution of differential equations arising in engineering problems is usually based on finite difference, finite element, boundary element or finite volume techniques. Other numerical methods like Direct Discrete Method (DDM) (Ayyoubzadeh et al., 2012) and (Vosoughi et al., 2003) may also be used to solve the specific problems. In general, the Finite Element Methods (FEMs) is preferred in most applications to its principal alternative, the Finite Difference Method (FDM), due to its flexibility in the treatment of curved or irregular geometries and the high rates of convergence attainable by the use of high order elements. The first application of FEM to the theory of neutron diffusion dates back to 1970s (Kang and Hansen, 1973). The development in the application of FEM to the neutron diffusion equation has been described in the excellent treatise of Lewis (Lewis, 1981). Recently, several other applications of FEMs including Raviart-Thomas-Schneider, Hybrid, h-adaptivity, Response Matrix, etc. to solve neutron diffusion equation has been introduced (Hébert, 2008), (Cavdar and Ozgener, 2004) and (Wang et al., 2009).

In the current study, the Galerkin FEM (GFEM), a weighted residual method, is used to solve the multi-group neutron/adjoint diffusion equation for hexagonal and rectangular reactor cores. For several reasons, such as desired precision, yet being simple, the Galerkin method has been widely used in the development of computer codes for solving the diffusion or transport equation in different geometries (Maiani and Montagnini, 2004). The main advantage of GFEM is that the definition of boundary conditions in this method is easier than the other methods (Zhu et al., 2005), (Hosseini and Vosoughi, 2013).

In this paper, the neutron/adjoint flux and neutron multiplication factor are obtained from multi-group neutron/adjoint equation using developed GFEM-2D and GELES computer codes. Unstructured triangular finite elements generated by the Gambit software (Gambit, 2006) are used to discretize the equations. Indeed, a key advantage of the unstructured triangular elements is their superiority in mapping the curved boundaries or material interfaces.

An outline of the remainder of this contribution is as follows: In

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**Abbreviations**

| | |
|---|---|
| FEM | finite element method, |
| $\Phi_g(r)$ | neutron flux in energy group g, |
| $k_{eff}$ | neutron multiplication factor, |
| $\chi_g$ | neutron spectrum in energy group g, |
| $\phi_g^{\dagger}$ | adjoint flux in energy group g, |
| $D_g$ | diffusion constant in energy group g, |
| $\Sigma_{a,g}$ | macroscopic absorption cross section in energy group g, |
| $\Sigma_{r,g}$ | macroscopic removal cross section in energy group g, |
| $\Sigma_{f,g}$ | macroscopic fission cross section in energy group g, |
| $\Sigma_{h\to g}$ | macroscopic scattering cross section from energy group *h* to *g*, |
| $\upsilon$ | fission neutron yield, |
| $\nabla$ | the nabla operator. |

Section 2, we briefly introduce the mathematical formulation used to solve the neutron/adjoint diffusion equation. Two approaches based on Galerkin and Generalized Least Squares Finite Element Methods (FEMs) are presented in the Section 2. Section 3 presents the main specification of the BIBLIS-2D (Varin et al., 2004), IAEA-2D (Center, 1977), VVER440-2D (Chao and Shatilla, 1995) and seven energy groups (Center, 1977) benchmark problems. The results obtained from GFEM-2D (developed computer code based on Galerkin FEM) and GELES-2D (developed computer code based on Generalized Least Squares FEM) are presented in Section 4. In Section 5, we discuss the results obtained from the GFEM-2D and GELES-2D computer codes and advantages of applying the unstructured triangular elements. Section 6 gives a summary and concludes the paper.

## 2. Mathematical formulation

### 2.1. Discretization of the neutron diffusion equation

#### 2.1.1. Galerkin finite element method

In the absence of external neutron source, the multi-group neutron diffusion equation is as Eq. (1) (Duderstadt and Hamilton, 1976), (Lamarsh, 1966):

$$-\nabla\cdot D_g(r)\nabla\Phi_g(r)+\Sigma_{r,g}(r)\Phi_g(r)=\frac{\chi_g}{k_{eff}}\sum_{g'=1}^{G}\nu\Sigma_{f,g'}(r)\Phi_{g'}(r)+\sum_{h\neq g}\Sigma_{h\to g}(r)\Phi_h(r)\quad g=1,\ 2,\ \ldots,\ G \tag{1}$$

where, all quantities are defined as usual.

Eq. (1) is a linear partial differential equation which may be solved by different numerical methods. All of these methods transform the differential equation into a system of algebraic equations. Here, GFEM, a weighted residual method, is used to discretize the neutron diffusion equation. To start the discretization, the whole solution area is divided into the unstructured triangular elements as shown in Fig. 1. These elements have been generated using Gambit mesh generator. In the linear approximation of shape function, the neutron flux in each element could be considered as Eq. (2) (Zhu et al., 2005):

$$\Phi^{(e)}(x,y)=N_i^{(e)}(x,y)\Phi_i+N_j^{(e)}(x,y)\Phi_j+N_k^{(e)}(x,y)\Phi_k \tag{2}$$

where, $N_i^{(e)}$, $N_j^{(e)}$ and $N_k^{(e)}$ are the components of the shape function. In the linear approximation, these components are equal to the corresponding $L_n^{(e)}(x,y)$ as Eq. (3):

$$L_n^{(e)}(x,y)=\frac{a_n+b_nx+c_ny}{2\Delta^{(e)}}\qquad n=i,j,k \tag{3}$$

in which

$$a_i=x_jy_k-y_jx_k;\quad b_i=y_j-y_k;\quad c_i=x_k-x_j \tag{4}$$

$$a_j=x_ky_i-y_kx_i;\quad b_j=y_k-y_i;\quad c_j=x_i-x_k \tag{5}$$

$$a_k=x_iy_j-y_ix_j;\quad b_k=y_i-y_j;\quad c_k=x_j-x_i \tag{6}$$

and

$$2\Delta^{(e)}=\begin{vmatrix}1 & x_i & y_i\\ 1 & x_j & y_j\\ 1 & x_k & y_k\end{vmatrix} \tag{7}$$

The components of the shape function satisfy the criterion given in Eq. (8) at all points of the domain:

$$N_i^{(e)}(x,y)+N_j^{(e)}(x,y)+N_k^{(e)}(x,y)=1 \tag{8}$$

here, the weighting function is considered as Eq. (9) to apply the Galerkin FEM:

$$W(r)=\underline{N}(r) \tag{9}$$

Multiplying Eq. (1) by the weighting function and integrating the results over the solution space, Eq. (10) is obtained:

$$\iint_{\Omega}d\Omega W(r)\left(-D_g\nabla^2\Phi_g(r)+\Sigma_{rg}\Phi_g(r)-\frac{\chi_g}{k_{eff}}\sum_{g'=1}^{G}\nu\Sigma_{fg'}\Phi_{g'}(r)-\sum_{h\neq g}\Sigma_{h\to g}\Phi_h^{(n)}(r)\right)=0 \tag{10}$$

in above equation, the differential part may be transformed by applying the Green's theorem:

$$\iint_{\Omega}d\Omega W(r)\left(-D_g\nabla^2\Phi_g(r)\right)=\iint_{\Omega}d\Omega\nabla W(r)\cdot\nabla\Phi_g(r)-\iint_{\Omega}d\Omega\nabla\cdot\left(W(r)\nabla\Phi_g(r)\right)=\iint_{\Omega}d\Omega\nabla W(r)\cdot\nabla\Phi_g(r)-\int_{\partial\Omega}\overrightarrow{ds}\cdot W(r)\nabla\Phi_g(r)=\iint_{\Omega}d\Omega\nabla W(r)\cdot\nabla\Phi_g(r)-\int_{\partial\Omega}ds\,W(r)\frac{\partial\Phi_g(r)}{\partial n} \tag{11}$$