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A generalized strain energy density criterion for mixed mode fracture analysis in brittle and quasi-brittle materials

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In this paper, a modified mixed mode fracture model called the generalized strain energy density criterion is proposed. The criterion uses three key crack parameters including: the mode I and mode II stress intensity factors (K_I and K_{II}) and the T-stress for predicating the mixed mode fracture resistance in brittle and quasi-brittle materials. The main purpose of this paper is to show that the T-stress has a major role in mixed mode I/II brittle fracture when an energy-based criterion is employed. The theoretical results obtained from the generalized strain energy density criterion are compared with the experimental results reported in previous papers for semi-circular bend specimens made of PMMA and Harsin marble and also for cracked triangular specimens made of Neiriz marble. It is shown that the results predicted using the modified criterion are in significantly better agreement with the experimental results compared with the conventional minimum strain energy density criterion.

1. Introduction

Several failure criteria are available for predicating the onset of brittle fracture under mixed mode I/II loading. Often divided into two major categories of stress-based criteria and energy-based criteria, these fracture models are applicable to brittle and quasi brittle materials such as ceramics, rocks, glasses and brittle polymers. There are three classical criteria for mixed mode brittle fracture: the maximum tangential stress (MTS) [1], the maximum energy release rate [2], and the minimum strain energy density (SED) [3] where the former is a stress-based criterion and the other two are energy-based ones. These criteria make use of singular terms of William's series expansion in order to estimate the angle of fracture initiation and the onset of brittle fracture under mixed mode loading when the amount of plastic deformation around the crack tip is negligible.

In many practical cases, a significant discrepancy has been reported between theoretical predications by the conventional mixed mode fracture criteria and the experimental results. In recent years, some researchers have used the higher order terms of William's expansion, in particular the T-stress, for improving the theoretical predictions. As a constant stress parallel to the crack, the T-stress is independent of the distance from the crack tip. Williams and Ewing [4] were first who used the MTS criterion to study the effect of T-stress on mixed mode brittle fracture but only in the angled crack problem where closed form solutions are available for stress intensity factors and T-stress. Later, Smith et al. [5] proposed a generalized maximum tangential stress (GMTS) criterion based on three crack parameters K_I, K_{II} and T, and showed that the T-stress has an important role in mixed mode brittle fracture. The stress-based criterion proposed by Smith et al. [5] had a simple formulation and could be employed for any arbitrary mixed mode crack problem. Then, Ayatollahi et al. performed a large number of experiments on different brittle or quasi-brittle materials and showed that the generalized MTS criterion is able to provide significantly better estimates for the onset of mixed mode fracture in comparison with the conventional MTS criterion [6–13].

Among the energy-based fracture theories, the maximum energy release rate criterion received less attention due to complexities associated to it. However, a large number of researchers made use of the minimum strain energy density (SED) criterion suggested by Sih [14] or similar criteria developed based on the state of strain energy density around the crack tip. Sih [14] introduced a strain energy density factor S that can be used for predicting the direction of crack propagation and the onset of fracture in combined tensile and shear loading. The strain energy density factor S is related to the amount of strain energy density in a critical distance r_c from the crack tip. In this critical region, very large strains are generated and the material is damaged due to severe deformation,

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microcracks, dislocations, etc. According to the SED criterion, fracture initiates along an angle θ_0 where the strain energy density factor has its minimum value at the critical distance r_c. In addition, the onset of fracture occurs when the value of strain energy density factor along this angle and at the critical radius r_c reaches a critical value S_{cr} which is a material property. Based on the Sih's concept of SED factor, several other criteria were proposed by numerous researchers. Maximum dilatational strain energy density [15], minimum distortional strain energy density [16] and maximum tangential strain energy density [17] criteria are to name a few. Lazzarin and his co-researchers also published a series of interesting papers using a synthesis based on the strain energy density averaged over finite-size control volumes around stress concentrators, see for example [18–24]. In this criterion, the averaged value of the strain energy density over the control volume is used to predict the load-bearing capacity of cracked, U-notched, V-notched and keyhole notched specimens under mixed-mode I/II conditions. Through extensive experimental data obtained for different materials and various notch shapes, Lazzarin showed that his fracture model is able to predict the experimental results very well.

There are some other fracture models like the theory of critical distances [25–28] or cohesive zone model [29–30] which deal with stable and unstable crack growth in engineering structures. The static failure assessment of cracked and notched components under pure mode I and mixed mode I–II loading conditions can be performed by the theory of critical distances (TCD) [25–28]. This theory consists of a series of different approaches. For instance, TCD suggests that under pure mode I loading the mechanical failure in engineering components containing a crack occurs when the mean stress over a line ahead of the crack tip (line method) or the stress obtained at a certain distance from the crack tip (point method) attains their critical values. The same concept was extended by Susmel and Taylor [25] to mixed mode I–II loading as well.

Although the effects of T-stress on mixed mode fracture have been studied well by using the GMTS criterion (as a stress-based fracture model), a review of literature shows that no energy-based fracture model has been employed in the past to explore in detail the T-stress effects in mixed mode brittle fracture. In this paper, a generalized minimum strain energy density criterion is presented for estimating both the angle of fracture initiation and the mixed mode fracture resistance of brittle and quasi-brittle materials. The criterion takes into account both the singular terms and the T-stress terms related to the stresses and strains in the Williams' series expansion. The theoretical results obtained from the generalized SED criterion is then compared with the experimental results reported in previous papers for the semi-circular bend (SCB) specimens made of PMMA and marble and also for the cracked triangular specimen subjected to three-point bending made of Neiriz marble. It is shown that very good agreement exists between the experimental results and theoretical results predicted using the generalized SED criterion.

2. Generalized SED criterion

Williams [31] proposed an Airy stress function and derived the elastic stress field around the tip of a crack as infinite series expansions:

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{2\sqrt{r}}\left[A_1(3-\cos\theta)\cos\frac{\theta}{2}+A_2(3\cos\theta-1)\sin\frac{\theta}{2}\right]+4A_3\cos^2\theta+O(r^{1/2})\\
\sigma_{\theta\theta} &= \frac{1}{2\sqrt{r}}\left[A_1(1+\cos\theta)\cos\frac{\theta}{2}-A_2 3\sin\theta\cos\frac{\theta}{2}\right]+4A_3\sin^2\theta+O(r^{1/2})\\
\sigma_{r\theta} &= \frac{1}{2\sqrt{r}}[A_1\sin\theta+A_2(3\cos\theta-1)]\cos\frac{\theta}{2}-4A_3\sin\theta\cos\theta+O(r^{1/2})
\end{aligned} \tag{1}$$

where σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ are elastic stresses in the polar coordinate system and (r, θ) are the crack tip co-ordinates (see Fig. 1). The constant coefficients A_1, A_2 and A_3 are dependent on the geometry and loading conditions in cracked structures, and can be related to the mode I and mode II stress intensity factors (K_I and K_{II}) and the T-stress as follows:

$$K_I = \sqrt{2\pi}A_1 \quad K_{II} = \sqrt{2\pi}A_2 \quad T = 4A_3$$

The non-singular stress term (T-stress) is constant and independent of the distance r from the crack tip. The higher order terms $O(r^{1/2})$ vanish in the vicinity of the crack tip.

The strain energy density function dW/dV stored in an element dV can be written as

$$\begin{aligned}
dW/dV = &\left[\frac{1}{2E}\left(\sigma_{rr}^2+\sigma_{\theta\theta}^2+\sigma_{zz}^2\right)-\frac{\nu}{E}(\sigma_{rr}\sigma_{\theta\theta}+\sigma_{rr}\sigma_{zz}+\sigma_{zz}\sigma_{\theta\theta})\right.\\
&\left.+\frac{1}{2G}\left(\sigma_{r\theta}^2+\sigma_{rz}^2+\sigma_{z\theta}^2\right)\right]
\end{aligned} \tag{2}$$

where σ_{ij} $(i, j = r, \theta, z)$ are the stress components. E and G are the modulus of elasticity and the modulus of rigidity, respectively. It is known that $G = E/2(1+\vartheta)$ in which ϑ is the Poisson's ratio. The out-of-plane stress is written as $\sigma_{zz} = \vartheta(\sigma_{rr} + \sigma_{\theta\theta})$ for plane strain and $\sigma_{zz} = 0$ for plane stress. The strain energy density function can be simplified and expressed for the plane elasticity problems as:

$$dW/dV = \frac{1}{2G}\left[\frac{\kappa+1}{8}(\sigma_{rr}+\sigma_{\theta\theta})^2-\sigma_{rr}\sigma_{\theta\theta}+\sigma_{r\theta}^2\right] \tag{3}$$

where κ is an elastic constant that takes the value of $3-4\vartheta$ for the plane strain problems and $\frac{3-\vartheta}{1+\vartheta}$ for plane stress ones.

The strain energy density factor, S, which represents the strength of the elastic energy field in the vicinity of the crack tip is defined as:

$$S = r\frac{dW}{dV} \tag{4}$$

By replacing the stress components from Eqs. 1–3 into Eq. (4) and simplifying the obtained equation, one may derive:

$$\begin{aligned}
S = &A_1K_I^2 + A_2K_{II}^2 + 2A_3K_IK_{II} + 2A_4(\sqrt{2\pi r})K_IT\\
&+2A_5(\sqrt{2\pi r})K_{II}T + A_6(2\pi r)T^2
\end{aligned} \tag{5}$$

in which

$$\begin{aligned}
A_1 &= \frac{1}{16\pi G}[(\kappa-\cos\theta)(1+\cos\theta)]\\
A_2 &= \frac{1}{16\pi G}[\kappa(1-\cos\theta)+\cos\theta(1+3\cos\theta)]\\
A_3 &= \frac{1}{16\pi G}\sin\theta(2\cos\theta-(\kappa-1))\\
A_4 &= \frac{1}{16\pi G}\cos\frac{\theta}{2}[\cos(2\theta)-\cos\theta+(\kappa-1)]\\
A_5 &= \frac{-1}{16\pi G}\sin\frac{\theta}{2}[\cos(2\theta)+\cos\theta+(\kappa+1)]\\
A_6 &= \frac{1}{16\pi G}\left(\frac{\kappa+1}{2}\right)
\end{aligned} \tag{6}$$

According to the generalized SED criterion, the initial crack growth takes place in the direction θ_0 where the amount of the SED factor is minimum at a critical distance from the crack tip r_c. Therefore, the angle θ_0 is determined from:

$$\left.\frac{\partial S}{\partial\theta}\right|_{\theta=\theta_0} = 0 \quad \left.\frac{\partial^2 S}{\partial\theta^2}\right|_{\theta=\theta_0} > 0 \tag{7}$$

which gives

$$\begin{aligned}
&C_1K_I^2 + C_2K_{II}^2 + C_3K_IK_{II} + C_4(B\alpha K_{eff})K_I + C_5(B\alpha K_{eff})K_{II}\\
&+C_6(B\alpha K_{eff})^2 = 0
\end{aligned} \tag{8}$$