Theoretical and Applied Mechanics Letters 5 (2015) 107–110

Contents lists available at ScienceDirect

Theoretical and Applied Mechanics Letters

journal homepage: www.elsevier.com/locate/taml

Letter

WKBJ analysis in the periodic wake of a cylinder

F. Giannetti

Department of Industrial Engineering, University of Salerno, 84084 Fisciano (SA), Italy

ARTICLE INFO

Article history:
Received 31 July 2014
Received in revised form
15 November 2014
Accepted 11 December 2014
Available online 9 March 2015
*This article belongs to the Fluid Mechanics

Keywords:
Cylinder wake
Secondary instability
Lagrangian trajectories
WKBJ approximation

ABSTRACT

The nature of the three-dimensional transition arising in the flow past a cylinder is investigated by applying the Lifschitz–Hameiri theory along special Lagrangian trajectories existing in its wake. Results show that the von Kármán street is unstable with regard to short-wavelength perturbations. The asymptotic analysis predicts the possible existence of both synchronous (as modes A and B) and asynchronous (as mode C) instabilities, each associated to specific Lagrangian orbits. The proposed study provides useful qualitative information on the origin of the different modes but no quantitative agreement between the growth rates predicted by the asymptotic analysis and by a global stability analysis is observed. The reasons for such mismatch are briefly discussed and possible improvements to the present analysis are suggested.

The flow in the wake of a two-dimensional cylinder becomes first unstable to three-dimensional disturbances at a Reynolds number (based on the free-stream velocity and the cylinder diameter) $Re \approx 190$. Experiments and numerical simulations have highlighted the presence of two different shedding modes with different spatial characteristics, generally referred to as modes A and B (see for examples in Refs. [1,2]). Floquet analysis [3] certifies the existence of two separate bands of unstable modes: the first one (mode A) emerges for $Re > 189$ and has a spanwise wavelength of about 4 cylinder diameters, while the second one (mode B) appears for $Re > 259$ and is characterized by a shorter spanwise wavelength (about 0.8 diameter). Both of them are synchronous modes, i.e., they have the same periodicity of the base flow. An asynchronous quasi-periodic mode (usually termed mode C) with an intermediate wavelength also exists and was revealed by inserting in the flow a thin wire placed parallel to the cylinder axis (see Ref. [4]). Depending on the geometry this mode can be stable (as in the case of a circular cylinder) or unstable (square cylinder and other geometries). It is important to recall that the characteristics of the above mentioned modes and the associated transition scenarios are not specific to circular cylinders, but applies to a whole range of two-dimensional geometries ranging from square cylinders [5] to long plates with aerodynamic noses [6]. Despite the large number of experimental, theoretical and numerical studies performed on similar geometries, the precise nature of these modes is not fully understood yet. Several different mechanisms have been proposed to explain their genesis, including elliptic [7,8], hyperbolic [7], centrifugal [9], or Benjamin–Feir [10] instabilities (see for instance in Ref. [11] for a detailed discussion). However, no conclusive evidence supporting these speculations was given. In particular, a weak point of all these models consists in the fact that they are all based on idealized stationary flow configurations, while the real wake flow evolves in time and space in a complex way.

The scope of this letter is to illustrate an alternative approach to investigate the nature of the secondary instability which is based on the application of the Lifschitz and Hameiri theory [12,13] along particular orbits in the wake of the cylinder. The proposed analysis is an attempt to overcome the limitation of the previous theories bringing together results obtained through sensitivity analysis and asymptotic techniques.

Helpful information on the spatial and temporal evolution of the secondary instability can be retrieved by performing a structural sensitivity analysis of the unstable Floquet modes to localized force–velocity feedbacks, as proposed and explained in Refs. [14,15]. This procedure allows one to identify the instability core by inspecting the spatial structure of the instantaneous sensitivity tensor

$$\boldsymbol{I}(x,y,k,t) = \frac{\boldsymbol{f}^{+}(x,y,k,t)\,\boldsymbol{u}(x,y,k,t)}{\int_t^{t+T}\int_{\mathcal{D}}\boldsymbol{f}^{+}\cdot\boldsymbol{u}\,\mathrm{d}S\,\mathrm{d}t} \tag{1}$$

where $\boldsymbol{u}$ and $\boldsymbol{f}^{+}$ are respectively the direct and adjoint Floquet eigenvectors and k is the wavenumber in the periodic direction. By plotting its spectral norm, it is possible to trace the spatial and temporal evolution of the instability core during the phases of the vortex shedding. Results for mode A and mode B show that the instability is very localized in space and evolves in times in a com-

E-mail address: fgiannetti@unisa.it.

http://dx.doi.org/10.1016/j.taml.2015.03.001

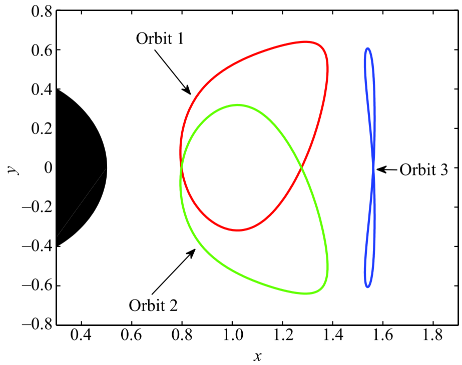

Fig. 1. Numbering of the closed orbits existing in the wake of a cylinder at $Re = 260$.

plex way (see Ref. [14] for details). Preliminary results obtained on a square cylinder at $Re = 205$ presents similar characteristics for all unstable modes. Moreover as noticed by Camarri and Giannetti [14], in the wake of the cylinder there exist closed Lagrangian trajectories, i.e., orbits described by material points which return to their initial position after a shedding cycle. Such orbits are solutions of the ordinary differential equation (o.d.e.)

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{U}_\mathrm{b}(\boldsymbol{X}(t), t) \tag{2}$$

with the periodic condition

$$\boldsymbol{X}(t+T) = \boldsymbol{X}(t). \tag{3}$$

In these expressions $\boldsymbol{X} \equiv \{x_o, y_o\}$ indicates the coordinates of the material points lying along the Lagrangian trajectory, $\boldsymbol{U}_\mathrm{b}$ is the periodic base flow and T is the shedding period. Here we used a fourth-order Runge–Kutta scheme coupled to a Newton–Raphson procedure to solve Eqs. (2) and (3), while the base flow was determined with the same finite-difference immersed boundary code described in Ref. [14]. The analysis was performed for both circular and square cylinders (not documented here for sake of brevity): in both cases we found three closed Lagrangian trajectories with shapes and symmetries similar to those depicted in Fig. 1. For the sake of precision, we have to say that there are actually an infinity of closed orbits satisfying Eq. (2) but on which a material point comes back to its initial position after $n > 1$ shedding periods. These solutions, however, are not considered in the present letter and are left for future investigations. The sensitivity analysis reveals that the instability core for modes A and B is highly localized in space: the highest sensitivity regions strictly follow the position of the material points moving along the closed orbits, showing the existence of a strong correlation between these orbits and the instability core. An example of such behavior is reported in Fig. 2, where the temporal evolution of the points on the closed orbits are depicted for mode B (at $Re = 260$) together with the spectral norm of the sensitivity tensor $\boldsymbol{I}(x, y, k, t)$ during a whole shedding period. Similar conclusions hold for the unstable modes arising in the wake of a square cylinder.

The strong localization of the instability core revealed by the structural sensitivity analysis suggests the possibility to use a "local theory" to describe its generation and evolution. An appealing approach in this context is given by the short-wavelength (WKBJ) approximation introduced by Lifschitz and Hameiri [12,13]. Here, the solution of the linearized Navier–Stokes equations is sought in the form of a rapidly oscillating and localized wave-packet evolving along a Lagrangian trajectory $\boldsymbol{X}(t)$ and characterized by a wavevector $\boldsymbol{k}(t) = \nabla\phi(\boldsymbol{x}, t)$ and an envelope $\boldsymbol{a}(\boldsymbol{x}, t)$ such that

$$\{\boldsymbol{u}, p\}(\boldsymbol{x}, t) = \boldsymbol{a}(\boldsymbol{x}, t) \exp(\mathrm{i}\,\phi(\boldsymbol{x}, t)/\epsilon) \tag{4}$$

Table 1
Eigenvalues of the fundamental Floquet matrix associated to Eq. (5) for the three closed orbits. See Fig. 1 for the orbit numbering.

Orbit	Re	μ_1	μ_2	μ_3
1,2	190	−0.0317	−31.6541	1
3	190	+0.0127	+78.9224	1
1,2	260	−0.0138	−72.8052	1
3	260	+0.0055	+181.2131	1

with $\epsilon \ll 1$. In the limit of vanishing viscosity ($Re \to 0$) and large wavenumbers ($\|\boldsymbol{k}\| \to \infty$), the theory enables one to evaluate, at leading order, the growth rate associated with a localized perturbation. This is achieved by solving the following set of linear o.d.e.

$$\frac{\mathrm{D}\boldsymbol{k}}{\mathrm{D}t} = -\mathcal{L}^t(\boldsymbol{X})\boldsymbol{k}, \tag{5}$$

$$\frac{\mathrm{D}\boldsymbol{a}}{\mathrm{D}t} = \left(\frac{2\boldsymbol{k}\boldsymbol{k}^t}{|\boldsymbol{k}|^2} - \mathcal{I}\right)\mathcal{L}(\boldsymbol{X})\boldsymbol{a} \tag{6}$$

along the Lagrangian trajectories satisfying Eq. (2) with some initial conditions. In the previous equations $\mathcal{L} = \nabla\boldsymbol{U}_\mathrm{b}$ is the velocity gradient tensor of the base flow, $\mathcal{I}$ is the identity tensor and the superscript "t" indicates the transpose operator.

As proved by Lifschitz and Hameir [12,13], inviscid instability occurs when such system has at least one solution with $\|\boldsymbol{a}(t)\| \to \infty$ as $t \to \infty$. This theory has been successfully applied in the past to study centrifugal, elliptic and hyperbolic instabilities developing on 2D steady base flows (see for examples [16–22]). In order to characterize the instability mechanism occurring in the periodic wake of the cylinder using such local theory, however, the self-excited nature of the instability must be properly accounted for. In such context, a central role is played by the closed Lagrangian trajectories described in the previous section. Such trajectories might play a special role in the dynamics of the instability: from an inviscid point of view, in fact, local instability waves might propagate on the closed orbits and feedback on themselves leading to a self-excited mode.

In order to apply the theory, both Eqs. (5) and (6) must be integrated along the three closed trajectories found in the wake. Since the base flow is periodic, Eq. (5) is a linear o.d.e. with periodic coefficients whose general solution can be written in terms of Floquet modes. In particular, the solution can be found by building the fundamental Floquet matrix $\mathcal{M}(T)$, solution of the system

$$\frac{\mathrm{D}\mathcal{M}}{\mathrm{D}t} = -\mathcal{L}^t(\boldsymbol{X})\mathcal{M}, \tag{7}$$

$$\mathcal{M}(0) = \mathcal{I}, \tag{8}$$

and extracting its eigenvalues and the corresponding eigenvectors. Using these eigenvectors as initial conditions to integrate Eq. (5), it is possible to retrieve the temporal evolution of $\boldsymbol{k}$ during a whole shedding cycle. Equation (5) admits three independent solutions related to the three eigenvectors of the fundamental Floquet matrix $\mathcal{M}(T)$. The corresponding eigenvalues μ for both $Re = 190$ and $Re = 260$ are listed in Table 1. Since the base flow is periodic and 2D, for each orbit there exists an eigenvalue equal to 1 with a corresponding eigenvector which remains constant in time and perpendicular to the base flow. The other two eigenvectors, instead, lie in the same plane as the base flow and are associated with a complex conjugate pair of eigenvalues.

Once Eq. (5) is solved, the amplitude $\boldsymbol{a}$ can be found by integrating Eq. (6). In principle, to set $\boldsymbol{k}$ in Eq. (6), we can use a general linear combination of the Floquet modes previously determined. However, since we are trying to determine a self-excited mode, we only consider solutions of Eq. (5) which are periodic in time, i.e., solutions such that $\boldsymbol{k}(0) = \boldsymbol{k}(T)$. Therefore, only the constant eigenvector orthogonal to the base flow $\boldsymbol{k} = k\hat{\boldsymbol{z}}$ (associated