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Mean and dispersion of stress tensors using Euclidean and Riemannian approaches

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ABSTRACT

Stress is central to many aspects of rock mechanics, and in the analysis of *in situ* stress measurement data the calculation of the mean value and an assessment of dispersion are important for statistical characterisation. Currently, stress magnitude and orientation are processed separately in such analyses. This effectively decomposes the second-order stress tensor into scalar (principal stress magnitudes) and vector (principal stress orientations) components, and calculation of mean and dispersion of stress data on the basis of these decomposed components, which violates the tensorial nature of stress, may either yield biased results or be difficult to conduct. Here, by introducing tensorial techniques, we examine two calculation approaches for the mean and dispersion for stress tensors – based on Euclidean and Riemannian geometries – and discuss their similarities, differences and potential applicability in engineering practice. We compare the two approaches using stress tensor superposition and interpolation, and the analysis of actual *in situ* stress data. The results indicate that Euclidean and Riemannian mean tensors are in general not equal, with the disparity increasing as stress tensor dispersion increases. Both Euclidean and Riemannian approaches are shown to be capable of characterising stress dispersion, although Euclidean dispersion is scale dependent and has units of stress whereas Riemannian dispersion is a scale independent unitless number. Finally, a paradox is revealed in that despite stress tensors being Riemannian entities, it is Euclidean mean stress that is the more meaningful for engineering applications.

1. Introduction

Stress is central to many aspects of rock mechanics, and in the analysis of *in situ* stress measurement data the calculation of the mean value and an assessment of dispersion are important for statistical characterisation.[8,12,13,32–36] Currently, stress magnitude and orientation are customarily processed separately in such analyses (Fig. 1).[1–8,32–34,36] This effectively decomposes the second-order stress tensor into scalar (principal stress magnitudes) and vector (principal stress orientations) components, and calculation of mean and dispersion of stress data on the basis of these decomposed components, which violates the tensorial nature of stress, may either yield biased results or be difficult to conduct[9,10,13,37 (p54)]. As noted elsewhere,[11] '*Since stress is a tensor with six independent components, calculating the mean, standard deviation and confidence intervals of the measured stresses cannot be carried out using the same statistical techniques developed for scalar quantities*'. As an alternative to the separate analysis of principal stress magnitude and orientation, several researchers in the field of rock mechanics have calculated the mean stress tensor based on tensors referred to a common Cartesian coordinate system.[11–14,35] Although these contributions essentially introduced a tensorial approach, they did so in an empirical setting. A result of this is that, to date, there seems to have been no mathematically rigorous proposal from the rock mechanics community for calculating such summary statistics for groups of stress tensors as the mean and dispersion. In particular, the calculation of the dispersion of a group of stress tensors obtained from a stress measurement campaign seems not to have been conducted in the rock mechanics field. Here, continuing the analysis of stress tensors referred to a common Cartesian coordinate system, and considering tensors as single entities, we introduce approaches based on Euclidean and Riemannian geometry to calculate their mean and dispersion.

As an early tensorial application example in rock mechanics, Hyett et al.[12] demonstrated that the mean of n stress tensors should be found by firstly transforming the individual tensors to a common Cartesian coordinate system (say, x–y–z), and then calculating the mean of each tensor component:

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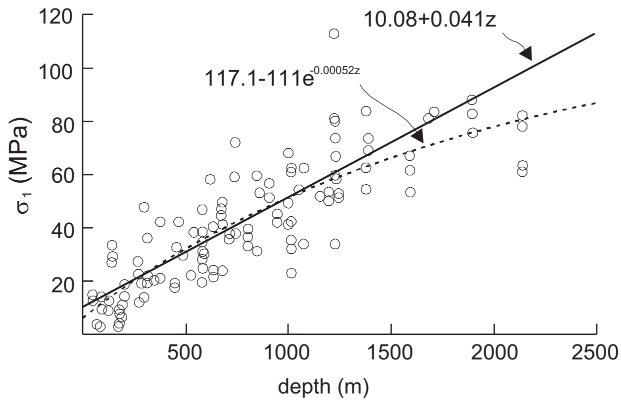
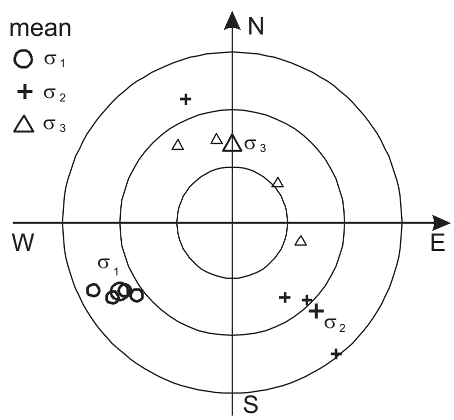


(a) Least squares regression of stress magnitude



(b) Directional statistics applied to principal stress orientation

Fig. 1. Example separate analyses of stress magnitude and orientation.[1] (a) Least squares regression of stress magnitude. (b) Directional statistics applied to principal stress orientation.

$$\bar{\mathbf{S}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{S}_i = \begin{bmatrix} \bar{\sigma}_x & \bar{\tau}_{xy} & \bar{\tau}_{xz} \\ & \bar{\sigma}_y & \bar{\tau}_{yx} \\ symmetric & & \bar{\sigma}_z \end{bmatrix} = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n}\sigma_{x_i} & \frac{1}{n}\sum_{i=1}^{n}\tau_{xy_i} & \frac{1}{n}\sum_{i=1}^{n}\tau_{xz_i} \\ & \frac{1}{n}\sum_{i=1}^{n}\sigma_{y_i} & \frac{1}{n}\sum_{i=1}^{n}\tau_{yz_i} \\ symmetric & & \frac{1}{n}\sum_{i=1}^{n}\sigma_{z_i} \end{bmatrix}. \tag{1}$$

Here $\bar{\mathbf{S}}$ denotes the mean stress tensor, $\mathbf{S}_i$ represents a particular stress tensor, σ and τ are the normal and shear tensor components, respectively, and $\bar{\sigma}$ and $\bar{\tau}$ denote the corresponding mean tensor components. This approach was subsequently followed by others.[11,13,14,35]

Based on Eq. (1), several researchers[11,13,35,39] suggested how the variance of stress tensors might be calculated. After obtaining the mean stress tensor, a new coordinate system is established that coincides with the principal directions of the mean tensor (say, *X–Y–Z*), and all the original stress tensors transformed into this new coordinate system. Using the fundamental definition of variance, i.e. $\sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)$, and recognising that $\bar{\tau}_{XY}=\bar{\tau}_{YZ}=\bar{\tau}_{ZX}=0$, the variance tensor is then calculated as

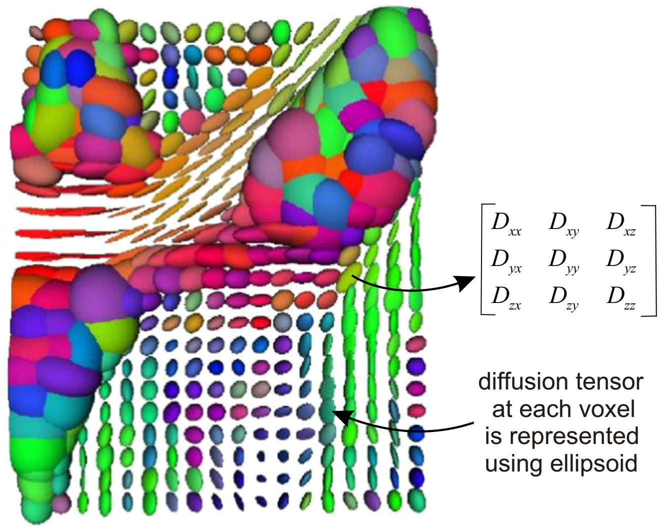


Fig. 2. Diffusion tensor at each voxel in Diffusion Tensor Imaging.[17]

$$\sigma_{\mathbf{S}}^2 = \frac{1}{n-1}\begin{bmatrix} \sum_{i=1}^{n}\left(\sigma_{X_i}-\bar{\sigma}_X\right)^2 & \sum_{i=1}^{n}\left(\tau_{XY_i}\right)^2 & \sum_{i=1}^{n}\left(\tau_{XZ_i}\right)^2 \\ & \sum_{i=1}^{n}\left(\sigma_{Y_i}-\bar{\sigma}_Y\right)^2 & \sum_{i=1}^{n}\left(\tau_{YZ_i}\right)^2 \\ symmetric & & \sum_{i=1}^{n}\left(\sigma_{Z_i}-\bar{\sigma}_Z\right)^2 \end{bmatrix}. \tag{2}$$

However, this only gives the dispersion of each tensor component, rather than a scalar value indicating the overall variability of the group of tensors. In other words, comparison of the dispersions of different groups of tensors is still difficult to conduct with this approach.

As we show below, these rock mechanics tensorial applications are essentially a Euclidean approach. However, it is now known that symmetric positive definite (SPD) matrices such as stress tensors with positive principal stresses do not live in Euclidean space, but in curved spaces known as Riemannian manifolds (see for example, Chapter 6 of Ref. 15 and Chapter 19 of Ref. 16). Statistical analysis of SPD matrices on Riemannian manifolds has been recently developed for use in Magnetic Resonance Imaging (MRI) applications in medicine (Fig. 2). MRI can be used to detect diffusion of water molecules through biological tissues, and analysis of this can reveal microscopic details about tissue architecture (either normal or in a diseased state). As diffusion can be characterised by an SPD matrix called the “diffusion tensor”, it has been necessary to develop tensorial approaches to aid diagnosis.[17–19]

In this paper, and following on from earlier work of ours,[20] we focus on the illustration and comparison of Euclidean and Riemannian approaches to calculating the mean and dispersion of stress tensors, and their potential applicability in engineering practice. The underlying stochastic model is one that simultaneously includes all stress tensor components (when referred to a common Cartesian coordinate system), rather than one that processes principal stress magnitudes and orientations separately. Since these calculations are based on distance measures, we first give a simple comparison of Euclidean and Riemannian distances, and indicate their significance in the calculation of mean tensors. Following this we introduce both Euclidean and Riemannian mean and dispersion functions, and present tensor superposition techniques in both Euclidean and Riemannian spaces. We move on to compare Euclidean and Riemannian approaches through stress tensor superposition and interpolation, and the analysis of actual and perturbed *in situ* stress data. We conclude by examining the differences between the two approaches and their respective