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Analytical solutions for rock stress around square tunnels using complex variable theory

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ABSTRACT

The complex variable theory is employed to find the analytical solution for rock stress around square tunnels in a homogeneous, isotropic and elastic rock mass. The solution is more accurate than previous available solutions, because the first three terms of transformation function are taken in the early deduction. We find that in situ stress and coefficients of lateral pressure play a crucial role in stress distribution. High compressive stress concentrations are found to exist at the four square corners. The surrounding rock is compressed over the complete square periphery when the pressure coefficients with values between 0.8 and 1.2. The boundary stress is gradually converted from tensile stress to compressive stress for the two sidewalls with the increasing pressure coefficient, whereas the opposite situation occurs for the roof and floor. In order to avoid the stress concentration, a square tunnel should choose a rounded corner and take certain support reinforcement measures. Besides, the results provide a theoretical basis on support design for deep square tunnels, and a universal framework to analyze surrounding rock stability for other tunnels of noncircular shape.

1. Introduction

Structural stability of tunnels is an essential aspect of rock engineering, due to the deformation and failure of the surrounding rock, with the result that the calculation of boundary stress is of significant interest to the stability analysis [1–3]. Although the analytical solutions for stress distribution around circular and elliptical holes have been given by many authors [4,5], it has not yet come to an available expression for noncircular cross sections such as square, rectangular, trapezoid and semicircular arched which we are often found in underground and mining engineering.

Because this kind of issue is of great theoretical significance and practical value in engineering applications, there developed several methods in the last century including traditional analysis, classical numerical simulation, and modern numerical simulation. In traditional analysis method, Muskhelishvili [6] fully expounded a basic theory of complex variable function in order to address some issues of plane elasticity mechanics, resulting in many complex problems of holes solved. Then Savin [7] and Lekhnitskii [8] introduced the complex variable method to stress concentration around holes for an anisotropic elastic body. Other contributions to the complex variable formalism were later made by England [9].

Much effort has been directed toward the study on stress distribution of noncircular shapes by means of complex functions and conformal mapping. Exadaktylos and Stavropoulou [10] presented a closed-form plane strain solution for stresses and displacements around semicircular tunnels, also compared the results with numerical models. Li and Wang [11] investigated the influences of the relative rigidity and thickness on the stress fields of a lined tunnel, drawing the conclusion that the factors should be in an appropriate range. Kargar et al. [12] calculated the stress components around a lined non-circular tunnel, demonstrating that a semi-analytical solution is much faster with high accuracy than numerical solution in predicting the results. Lu et al. [13] took support delay into consideration and obtained the distribution of tangential stresses surrounding the rock mass of a non-circular tunnel. Lei [14] and Zhao [15] et al. made an attempt to study rock stress around rectangular holes, also discussed the effects of span-depth ratio on stress distribution. Huo [16] and Shi [17] et al. further obtained a general equation for deep rectangular excavation with a far-field shear stress, as well as studied the relevant influencing factors, such as Poisson's ratio and structure size. Ekneligoda and Zimmerman [18,19] derived more general closed-form solutions for hole compressibility and shear compliance of regular and quasi-polygons which obtained from more than two terms of the mapping function, indicating that the complex variable theory is indeed a powerful method.

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Based on the research findings proposed above, this paper presents an analytical solution for square tunnels. First the basic theory of complex variables is systematically described with a series of logical formulae. Accordingly, the expression of boundary stress for square tunnels is finally obtained through the deduction processes step by step. An engineering example is also provided to illustrate how the analytical solution can be used to infer the state of in situ stress, reaching the conclusion that stress distribution of square tunnels is definitively affected by in situ stress and coefficients of lateral pressure.

2. Basic theory of complex variables

2.1. Fundamental formulae

The complex variables method is effective to analyze the stress concentration of plane holes, and it has obvious advantages in solving problems associated with multiply connected region, complex geometry shapes and high stress gradient. In order to derive the stress expressions of a single hole in indefinite elastic body, conformal mapping is used to transform the occupied area of elastic body in z-plane (e.g. xy-plane) to the center of the unit circle in ζ-plane.

According to the model with symmetric structure and loading, every position surrounding the square hole can be mapped to the unit circle, as shown in Fig. 1(a) and (b). And various positions of tunnel boundary are represented by a sequential list of numbers in Fig. 1(b), where point 1 represents the tunnel sidewall, point 4 represents the tunnel corner, point 7 represents the tunnel roof.

Based on Muskhelishvili's complex elastic theory, there must be a stress function if body force is a constant value in plane stress problems. This biharmonic function of position coordinates can always be expressed as

$$\varphi(\zeta) = \frac{\mu+1}{8\pi}(X+iY)\ln\zeta + B\omega(\zeta) + \varphi_0(\zeta)$$
$$\varphi_0(\zeta) = \alpha_1\zeta + \alpha_2\zeta^2 + \alpha_3\zeta^3 + \cdots + \alpha_n\zeta^n \quad (1)$$

$$\psi(\zeta) = \frac{\mu-3}{8\pi}(X-iY)\ln\zeta + (B'+iC')\omega(\zeta) + \psi_0(\zeta)$$
$$\psi_0(\zeta) = \beta_1\zeta + \beta_2\zeta^2 + \beta_3\zeta^3 + \cdots + \beta_n\zeta^n \quad (2)$$

The general formula of transformation function is given as

$$z = \omega(\zeta) = R\left(\frac{1}{\zeta} + \sum_{k=0}^{n} c_k\zeta^k\right) = R\left(\frac{1}{\zeta} + c_0 + c_1\zeta + c_2\zeta^2 + \cdots + c_n\zeta^n\right) \quad (3)$$

where n is a finite positive integer, R is a real number reflecting the hole's size, and the c_k are generally complex numbers. In most cases it is accurate enough to only take the first two or three terms of the series. In ζ-plane boundary, we assume ρ and θ are two polar coordinates of point ζ, set $\rho=1$ and $\zeta=\rho\,(\cos\theta + i\sin\theta) = \rho e^{i\theta} = e^{i\theta}$, also introduce a symbol σ ($\sigma=\zeta=e^{i\theta}$).

If it is assumed that f_0 is a known function of σ, f_0 is given by the expression

$$f_0 = i\int\left(\overline{X}+i\overline{Y}\right)ds - \frac{X-iY}{2\pi}\ln\sigma - \frac{1+\mu}{8\pi}(X-iY)\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\sigma - 2B\omega(\sigma) - \left(B'-iC'\right)\overline{\omega(\sigma)} \quad (4)$$

where X and Y are total surface stresses of square boundary along x- and y-directions, while B, $B'+iC'$ and $B'-iC'$ are associated with far field principal stress.

By means of Cauchy integral formula, the expression of $\varphi_0(\zeta)$ can be concluded as

$$\varphi_0(\zeta) + \frac{1}{2\pi i}\int_\sigma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\varphi_0'(\sigma)}}{\sigma-\zeta}d\sigma = \frac{1}{2\pi i}\int_\sigma \frac{f_0}{\sigma-\zeta}d\sigma \quad (5)$$

Substitution for Eqs. (4) and (5) in Eq. (6), gives the expression of $\psi_0(\zeta)$:

$$\psi_0(\zeta) + \frac{1}{2\pi i}\int_\sigma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\varphi'_0(\sigma)}{\sigma-\zeta}d\sigma = \frac{1}{2\pi i}\int_\sigma \frac{\bar{f}_0}{\sigma-\zeta}d\sigma \quad (6)$$

Furthermore, substitution for $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ in Eqs. (1) and (2), gives expressions of $\varphi(\zeta)$ and $\psi(\zeta)$ which can be simplified to

$$\Phi(\zeta) = \frac{\varphi'(\zeta)}{\omega'(\zeta)},\quad \Psi(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)} \quad (7)$$

where $\Phi(\zeta)$ and $\Psi(\zeta)$ are complex potential functions.

Using complex variable function to express the stress components in curvilinear coordinates, gives the equation

$$\sigma_\theta + \sigma_\rho = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}] = 4\mathrm{Re}\Phi(\zeta)$$

$$\sigma_\theta - \sigma_\rho + 2i\tau_{\rho\theta} = \frac{2\zeta^2}{\rho^2\overline{\omega'(\zeta)}}[\overline{\omega(\zeta)}\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)] \quad (8)$$

where σ_θ, σ_ρ and $\tau_{\rho\theta}$ are the stress components of the elastic body in curvilinear coordinates.

2.2. Theoretical framework

According to the basic method of complex elastic theory, there exist some certain steps to follow for solving rock stress problems of any kind of complex shapes of holes. It starts from a transformation function which can be found in relevant books [20] or

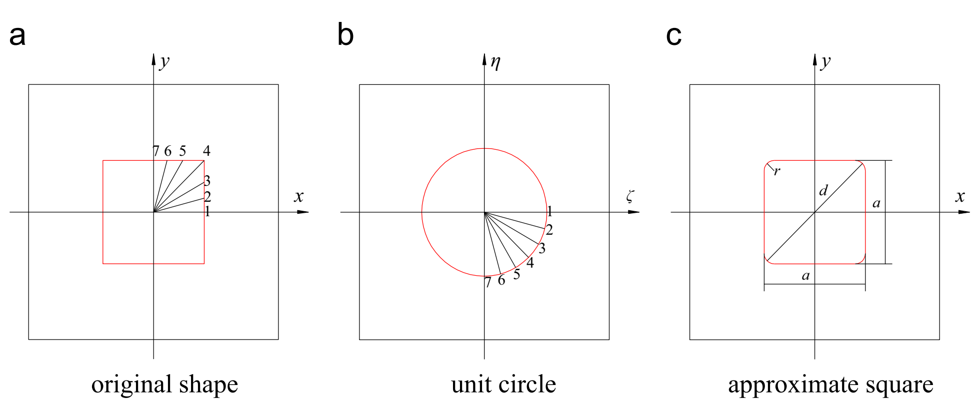

Fig. 1. Conformal mapping of square in z-plane to the center of the unit circle in ζ-plane.