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Boundary element analysis of non-planar three-dimensional cracks using complex variables

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This paper reports new developments on the complex variables boundary element approach for solving three-dimensional problems of cracks in elastic media. These developments include implementation of higher order polynomial approximations for the boundary displacement discontinuities and more efficient analytical techniques for evaluation of integrals. The approach employs planar triangular boundary elements and is based on the integral representations written in a local coordinate system of an element. In-plane components of the fields involved in the representations are separated and arranged in certain complex combinations. The Cauchy–Pompeiu formula is used to reduce the integrals over the element to those over its contour and evaluate the latter integrals analytically. The system of linear algebraic equations to find the unknown boundary displacement discontinuities is set up via collocation. Several illustrative numerical examples involving a single (penny-shaped) crack and multiple (semi-cylindrical) cracks are presented.

1. Introduction

Understanding the mechanisms of initiation and propagation of fractures in rock is of key importance to the mining and petroleum industries. The boundary element method (BEM) is an attractive tool for geomechanical applications as it is capable of efficient treatment of problems featuring large domains containing fractures (described as surfaces of discontinuous displacements) of arbitrary shapes [1,2]. In hydraulic fracturing simulations, the boundary equation for the crack is a fundamental part of a fully coupled model that involves fluid flow [3]. Therefore, accurate and efficient three-dimensional simulators of the fields around multiple non-planar cracks are in great demand.

In our previous publication [4], it was suggested to utilize complex variables for numerical solution of three-dimensional crack problems. While some applications of complex variables in three-dimensional elasticity theory have been reported (e.g., [5–8]), these publications dealt with the solution of a narrow class of theoretical problems that did not involve numerical simulations. However, in [4] it was shown that complex variables possess some attractive numerical features in simplification of analytical integration and reduction in the number of integral terms, as compared to real variables-based approaches. In the present paper, we extend the complex variables BEM of [4] to incorporate higher order approximations of the displacement discontinuities.

The present complex variables approach is displacement discontinuity (DD)-based in its broader sense (see the literature review in [4,9]), as it involves integral representation for tractions in terms of displacement discontinuities on crack surfaces. As in [4], planar triangular elements are employed and in-plane components of tractions, displacement discontinuities, as well as geometric parameters are arranged in various complex combinations. The entries of the matrix of the system of linear algebraic equations, set up via collocation, are calculated analytically by performing analytical integration over the element. However, instead of using simple piecewise constant approximations of the unknowns, as in [4], here we employ polynomials up to second order. In addition, instead of using an iterative integration procedure, which does not fully utilize the advantages of complex variables, we use the integration technique reported in [10]. The technique is based on representations for complex functions that reduce the integrals over the area of the element to those over its contour. We show that all entries of the influence matrix can be composed of various combinations of one generic integral, its derivatives, and their complex conjugates. The same integrals and combinations are involved in the representations of the elastic fields everywhere in the computational domain. The analytical expression for the generic integral over a triangular element is provided.

The goal of the present paper is to study the influence of higher order approximations on the accuracy of evaluation of the fields.

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While we do not introduce special crack tip shape functions for DD (as we are aiming at the applications, e.g. hydraulic fracturing, that do not necessarily exhibit square root asymptotic behavior), we show that higher order approximations still allow for capturing stress asymptotic behavior near the tips of dry cracks. The approach is tested on several examples, namely (i) the problem of a penny shaped crack under normal and shear far-field load, and (ii) the problem of two semi-cylindrical coaxial cracks under biaxial far-field load. Some of the results are tabulated to serve as benchmarks for future investigations.

2. Problem formulation and basic equations

We are concerned with the BEM solution of the problem of the stress state in an infinite elastic domain containing non-planar cracks of arbitrary shapes and subjected to far-field stresses (zero body force is assumed). The sign convention is that positive tensile stress.

BEM employs integral representations, which are equivalent to the governing partial differential equations of a specific problem, but expresses the solution of that problem in terms of integrals over the boundary of the domain of interest. The representation for the tractions $\mathbf{t}(\mathbf{x})$ at some point $\mathbf{x}$ located outside of the crack surfaces on the plane characterized by normal vector $\mathbf{n}(\mathbf{x})$ has the following form:

$$t_j(\mathbf{x}) = -\int_{\Sigma} H_{jk}(\mathbf{x},\boldsymbol{\xi})\Delta u_k(\boldsymbol{\xi})\, d\Sigma_{\xi} \tag{1}$$

where Σ is the totality of cracks' surfaces, $\boldsymbol{\xi} \in \Sigma$, Δu_k are the components of the displacement discontinuity (DD) $\boldsymbol{\Delta u} = \mathbf{u}^+ - \mathbf{u}^-$, in which $\mathbf{u}(\mathbf{x})$ is the displacement vector with the components u_k ($k = 1\ldots3$), "+" and "−" identify the displacements on the opposite sides of the crack, t_j are the components of the traction vector. The direction of the normal for the crack may be chosen arbitrarily. In the notations used, this direction indicates the outward normal for the "+" side (inward normal for the "−" side) of the crack.

Another notation adopted in Eq. (1) is related to the tensor $\mathbf{H}$ that represents the kernel of the hypersingular potential associated with Kelvin's fundamental solution [9,11–13]. Its components $H_{jk}(\mathbf{x},\boldsymbol{\xi})$ are given by the following expression:

$$\begin{aligned} H_{jk}(\mathbf{x},\boldsymbol{\xi}) = \frac{\mu}{4\pi(1-\nu)}\Big\{ & \frac{1-2\nu}{r^3}\left[n_m(\mathbf{x})n_m(\boldsymbol{\xi})\delta_{jk} + n_j(\boldsymbol{\xi})n_k(\mathbf{x})\right] \\ & - \frac{1-4\nu}{r^3} n_j(\mathbf{x})n_k(\boldsymbol{\xi}) + \frac{3(1-2\nu)}{r^5}\left[r_m n_m(\boldsymbol{\xi})n_j(\mathbf{x})r_k \right. \\ & \left. + r_m n_m(\mathbf{x}) r_j n_k(\boldsymbol{\xi})\right] \\ & + \frac{3\nu}{r^5} r_l n_l(\boldsymbol{\xi})\left[r_m n_m(\mathbf{x})\delta_{jk} + r_j n_k(\mathbf{x})\right] + \frac{3\nu}{r^5} r_k \left[n_m(\mathbf{x})n_m(\boldsymbol{\xi}) r_j \right. \\ & \left. + r_m n_m(\mathbf{x}) n_j(\xi)\right] - \frac{15}{r^7} r_m n_m(\mathbf{x}) r_l n_l(\xi) r_j r_k \Big\} \end{aligned} \tag{2}$$

where δ_{jk} is Kronecker's symbol, $r_k = x_k - \xi_k$, $r = |\mathbf{x} - \boldsymbol{\xi}|$, $n_k(\mathbf{x})$ are the components of $\mathbf{n}(\mathbf{x})$, $n_k(\boldsymbol{\xi})$ are the components of the outward unit normal vector $\mathbf{n}(\boldsymbol{\xi})$ at the point $\boldsymbol{\xi}$, μ is the shear modulus, and ν is Poisson's ratio.

The case of $\mathbf{x} \in \Sigma$ is handled as the limiting case in which the point $\mathbf{x}$ is allowed to reach the crack surface. Note that for the limit of the integral in Eq. (1) to exist, certain smoothness conditions are required for the boundary and the approximating functions for $\boldsymbol{\Delta u}$ ($\boldsymbol{\Delta u}(\boldsymbol{\xi}) \in C^{1,\alpha}$ [2,13]).

We assume that the tractions on Σ are prescribed. In case of far-field load, a standard superposition procedure is adopted, in which the problem is decomposed into two problems: one of an infinite domain (without cracks) under the far-field load and another one of an infinite domain containing the crack loaded by the prescribed corrective tractions and zero far-field stress. To find the approximate solution of the problem, the boundary is discretized, approximating (shape) functions for the unknowns are introduced, the integrals associated with each degree of freedom (and the corresponding shape function) are evaluated, and the system of linear algebraic equations is set by using the limit to the boundary and matching the prescribed boundary data (the standard steps of the BEM procedure). After the solution of the system of algebraic equations is found, the stresses inside the domain Ω can be reconstructed using the superposition of the prescribed far-field stress with the stresses given by the following discretized analog of Eq. (1) with appropriately chosen normal vectors $\mathbf{n}(\mathbf{x})$:

$$t_j(\mathbf{x}) = -\sum_{s=1}^{N_e}\int_{E_s} H_{jk}(\mathbf{x},\boldsymbol{\xi})\Delta u_k(\boldsymbol{\xi})\, d\Sigma_{\xi} \tag{3}$$

The integrals involved in (3) are regular integrals and evaluated using the same procedures as the ones used to assemble the matrix of the system of algebraic equations (influence matrix).

The displacements at the point $\mathbf{x}$ inside the domain of interest can be obtained (up to a rigid body movement) from the discretized analog of the following equation:

$$u_j(\mathbf{x}) = -\sum_{s=1}^{N_e}\int_{E_s} T_{jk}(\mathbf{x},\boldsymbol{\xi})\Delta u_k(\boldsymbol{\xi})\, d\Sigma_{\xi} \tag{4}$$

where the components T_{jk} of the tensor $\mathbf{T}$ that represents the kernel of the double layer potential are given in [4]. It should be noted that the formulation presented can be extended to the case of a finite domain containing cracks by including in Eqs. (1), (3) and (4) the corresponding integrals over the boundary of the finite domain (see [4]).

3. Assumptions and complex notations for the fields and geometry

As noted, in the present work planar triangular elements are used to discretize the cracks surfaces. The unknown displacement discontinuities on each element E_s are approximated by the linear combination of quadratic Lagrange polynomials associated with nodal points (6 per element). The positions of these points may vary; in the present paper two arrangements are studied: (a) nodes located on the contour of the element (including vertices) and (b) nodes located inside the element. The case (a) provides continuity of DDs across the edges, which the case (b) does not. Note that the collocation points can only be located inside the element due to the smoothness conditions required for the boundary and the approximating functions.

The local Cartesian coordinate system is introduced on the element E_s as in [4]: $\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3$ is the right-handed orthonormal basis where $\hat{\mathbf{e}}_1$ is parallel to the edge between two vertices of the element, $\hat{\mathbf{e}}_3 = -\mathbf{n}(\boldsymbol{\xi})$, and $\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1$. The origin of this coordinate system may be chosen arbitrarily. From now on, components of vectors and tensors are given in the local coordinate system of E_s (see Fig. 1).

The following combinations associated with the vector $\mathbf{r}$ that connects the point $\boldsymbol{\xi} \in E_s$ and the field point $\mathbf{x} = (x_1, x_2, x_3)$ are

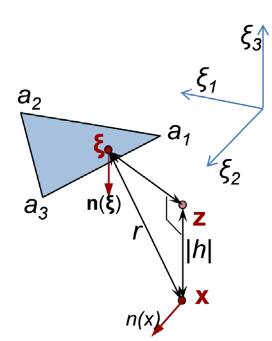

Fig. 1. Boundary element. Local coordinate system and notations.