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Applications of numerical limit analysis (NLA) to stability problems of rock and soil masses

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Abstract

This work presents formulations and results obtained with computer implementations of an alternative to the more standard techniques for the determination of the state of collapse of geotechnical structures in rock or soil masses. Examples of normally available and used techniques for those purposes are limit equilibrium based procedures and elasto-plastic finite elements. As an alternative to these techniques, the present paper describes Numerical Limit Analysis (NLA). The fundamentals for limit analysis, summarized in the so-called bound theorems, have been known for decades. Analytical solutions obtained with limit analysis are however limited in scope and are seldom used in the engineering practice. NLA on the other hand, by solving the limit analysis equations through numerical methods are general and applicable to a wide range of problems. The paper presents a discussion on available alternatives for the formulation of NLA specialized for the determination of collapse load factors of geotechnical structures in/on rock (fractured or not) and soil masses. Rock masses in particular are modelled as standard continua, Cosserat equivalent continua and true discontinua formed by discrete blocks. Finite elements are used for the solution of NLA equations of standard continua and Cosserat continua. The paper presents derivation of the pertinent equations, the numerical formulations used and details of their numerical implementation in computer programs. Attempt was made to validate all the implementations through existing analytical solutions. The obtained results permit to state that NLA is a promising and very often advantageous numerical technique to establish collapse states of geotechnical structures in rock and soil masses.

Keywords: Limit analysis; Numerical limit analysis; Fractured rocks; Discrete blocks; Cosserat continua; Standard continua

1. Introduction

In the design stage of a number of geotechnical problems such as the ones found in bearing capacity of foundations, retaining structures, slope stability and underground excavations, a primary objective consists in the determination of a collapse load, a maximum load the geotechnical system is able to support before it collapses. These loads are generally determined by limit equilibrium methods or elasto-plastic finite element analyses. In the present work, use is made of numerical limit analysis (NLA), an alternative, often very advantageous technique in relation to the ones described but in general seldom used in practice. The paper presents a theoretical background of limit analysis and possibilities of its numerical implementation. Main emphasis of the paper concerns applications of the technique for both continuum and discontinuum problems. In the latter case, more relevant to Rock Mechanics situations, the medium can be represented both by discrete blocks (true discontinua) and by Cosserat-type continua. Initially, the paper formulates the general limit analysis problem. Subsequently, it describes its specialization and numerical implementation for analysis of

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standard continuua, Cosserat continua and true discontinuua (discrete blocks) problems. Application and validation examples are presented and discussed. Finally, a general discussion on the possibilities of the use of NLA in practical problems is presented.

2. Fundamentals of limit analysis

The interest in the study of collapse of engineering structures began with Coulomb in 1773 [1]. Coulomb developed plasticity concepts applied to soils as well as the concept of a plastic limit as applied to retaining structures. Later, in 1857, Rankine introduced the concept of slip lines in order to interpret the plastic equilibrium of soils. Based in such studies, the concept of limit analysis evolved, more or less heuristically, until approximately 1950 in different areas of engineering. In 1952, Drucker and Prager, in a study of plastic materials obeying Mohr–Coulomb's yield criterion [2,3], defined bounds (upper and lower) for the collapse load.

The collapse load in stability problems can be obtained in an independent way through the application of the theorems of the limit analysis (upper bound or lower bound). When the collapse load is obtained from an established statically admissible stress field, it is considered to be a lower bound to the true collapse load. In the same way, if the collapse load is obtained by means of an established kinematically compatible failure mechanism, it is considered to be an upper bound to the true collapse load. In practice, the establishment of kinematically admissible failure mechanisms is in general easier when compared with the establishment of statically admissible stress fields. In this way, various analytical solutions are available today using various forms of simplified collapse mechanisms [4].

2.1. Limit analysis theorems

The application of the limit analysis theorems is possible for solid materials that present the following properties [1]:

1. The plastic behavior of the material is perfect or ideally plastic, i.e., the yield surface is fixed in the stress space.
2. The yield surface is convex and the rates of plastic deformation are obtained from a yield function through an associated flow rule.
3. Changes in the geometry of the body are considered insignificant when the loads reach the limit load state. The principle of virtual work can therefore be applied.

Central in the theory of limit analysis is the determination of a collapse load factor, roughly speaking an analogous measure to the factor of safety in limit equilibrium. In the lower bound formulation, the collapse load factor, or (or simply) the load factor can be defined as a multiplying factor (a scalar) by which the external loads have to be multiplied in order that the structural system reaches collapse. Similarly, in the upper bound formulation, the load factor is defined as a multiplying factor (also a scalar) by which the external work (work performed by the external loads) has to be multiplied in order that the structural system reaches collapse. Formally, the lower bound (or static) and the upper bound (kinematic) theorems can be stated as in the following:

Lower bound theorem (*static theorem*). A load factor λ_s that corresponds to a statically admissible stress field, one that satisfies (a) equilibrium equations in the domain, (b) equilibrium equations on the boundary and (c) nowhere in the domain the yield function is violated, will not exceed the true load factor of the structure.

Upper bound theorem (*kinematic theorem*). The kinematic load factor λ_k as determined by equating the rate of external work with the rate of internal dissipation of energy along a kinematically admissible velocity field ($\dot{\mathbf{u}}$), one that satisfies (a) the velocity boundary condition and (b) the compatibility relations between strain rates and velocity, is not less than the true collapse load factor.

According to the principles of continuum mechanics, the statements contained in both upper and lower bound theorems can be stated mathematically by the following equations:

Given

$$\mathbf{f} \quad \text{in } \Omega \text{ (body forces)},$$
$$\mathbf{t} \quad \text{on } \Gamma_t \text{ (boundary forces, tractions)},$$

Find λ, $\boldsymbol{\sigma}$, $\dot{\mathbf{u}}$, $\dot{\boldsymbol{\varepsilon}}$, e, $\dot{\boldsymbol{\gamma}}$ such that the following conditions are satisfied:

Static equilibrium:

$$\begin{aligned} \nabla^T \boldsymbol{\sigma} &= \lambda \mathbf{f} \quad \text{in } \Omega, \\ \boldsymbol{\sigma\eta} &= \lambda \mathbf{t} \quad \text{on } \Gamma_t, \end{aligned} \tag{1}$$

Yield criterion:

$$f(\boldsymbol{\sigma}) \leqslant 0 \quad \text{in } \Omega, \tag{2}$$

Kinematic consistency:

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}} &= \nabla \dot{\mathbf{u}} \quad \text{in } \Omega, \\ \dot{\mathbf{u}} &= 0 \quad \text{on } \Gamma_u, \end{aligned} \tag{3}$$

Flow rule:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma} \frac{\partial \mathbf{f}}{\partial \sigma} \quad \begin{aligned} \dot{\gamma} &= 0 \quad \text{if } f(\boldsymbol{\sigma}) < 0, \\ \dot{\gamma} &\geqslant 0 \quad \text{if } f(\boldsymbol{\sigma}) = 0, \end{aligned} \tag{4}$$

where $\mathbf{f}$ is the vector of body forces, $\mathbf{t}$ the boundary forces, tractions, $\boldsymbol{\eta}$ the unit vector normal to surface Γ_t, $\boldsymbol{\sigma}$ the stress field, $\dot{\mathbf{u}}$ the velocity (displacement rate) field, $\dot{\boldsymbol{\varepsilon}}^p$ the plastic strain rates field, λ the collapse load factor and $\dot{\gamma}$ the plastification factor.

The complete solution of the problem of establishing collapse of a system considering both statically and kinematically admissible fields must use Eqs. (1)–(4). The problem however can be solved by considering either a statically admissible field or a kinematically admissible field. When the collapse load of the system is obtained