Ultrasonics 91 (2019) 114–120

Contents lists available at ScienceDirect

Ultrasonics

journal homepage: www.elsevier.com/locate/ultras

On the validity of several previously published perturbation formulas for the acoustoelastic effect on Rayleigh waves

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ARTICLE INFO

Keywords:
Acoustoelastic effect
Rayleigh wave
Dispersion
Perturbation theory
Residual stresses

ABSTRACT

This article revisits the evaluation by a perturbation theory of the modification of the Rayleigh wave velocity under a static loading varying with depth. Two derivations, that have been exposed in the past and presented as comparable, are questioned. A new derivation of the perturbation formula is given by adapting Auld's approach. Validation with exact calculations is provided. The examples cover depth-varying static stress as well as depth-varying third order elastic properties.

1. Introduction

The slight modification of sound wave velocities when the propagation medium is statically stressed has been extensively studied in the past [1,2]. This effect is known as the acoustoelastic effect. The possibility of using it to monitor the state of residual stresses inside a material has been widely considered, and numerous applications have been developed in fields where either unwanted tensile stresses or deliberately generated compressive stresses play a major role on the lifetime of mechanical components. Previous works covered virtually all types of waves (bulk waves as well as surface or other guided waves). Because the strains involved are small, perturbation theory has been a dominant approach to predict the magnitude of the effect.

We shall in this work focus on the Rayleigh surface wave. This field has taken benefit from other communities which were already involved in studying the influence of depth-dependent texture on the dispersive character of the Rayleigh wave. A milestone was Auld's [3] perturbation theory, which lays on reciprocity relationships under a first order Born approximation (see Szabo [4] and Tittmann et al. [5] for early examples of application). The first work to deal with depth-varying loadings was probably that of Hirao et al. [6]. These authors used a perturbation approach to derive a formula which predicts a frequency-dependent behavior in the velocity of the Rayleigh wave, also providing experimental evidence in the case of a stress growing with depth. A few years later, Husson [7] addressed the same problem by using another way to derive the perturbation formula, based on an adaptation of Auld's methodology. Ditri and Hongerholt [8] later corrected typographical errors. Both articles of Hirao et al. and of Husson are today widely cited. Still, they do not agree.

This article is organized as follows. First, arguments are given to prove that neither the formula derived by Hirao et al. nor Husson's can cover arbitrary profiles of loading, and steps in both demonstrations referring to this fact are identified. Second, a new derivation of the perturbation formula is given by adapting Auld's approach. A general formula is given, and then applied to an initially isotropic half space. Finally, the several sets of formulas are compared numerically on diverse examples, together with a validation by an exact calculation.

2. Preliminary arguments

In what follows, ε_{ij}^{S} refers to the static strain, k to the wavenumber, x_1 is the coordinate in the direction of propagation, x_3 is the vertical coordinate and the over-bar means a value at the surface ($x_3 = 0$).

The formula derived by Hirao et al. expresses the variation of velocity of the Rayleigh wave Δv_R as a linear combination of $\bar{\varepsilon}_{ii}^{S}$, $\partial_3\bar{\varepsilon}_{ii}^{S}/k$, $\partial_3^2\bar{\varepsilon}_{ii}^{S}/k^2$, and integrals of $\varepsilon_{ii}^{S}(x_3)$ over the half-space weighted by decreasing exponentials. The formula derived by Husson has some common and some different features. It expresses the variation of velocity as a linear combination of $\bar{\varepsilon}_{11}^{S}$ and integrals of $\varepsilon_{ii}^{S}(x_3)$ over the half-space weighted by decreasing exponentials. In both cases, the presence of terms that explicitly depend on the value at the surface of ε_{ij}^{S} and its first two derivatives is problematic. Indeed, if we consider a loading which is located near the surface, *i.e.* which has a finite extent in depth, then the integral terms can be shown to tend to zero at low frequencies. The predicted low frequency behavior would then be of the form $\Delta v_R^{(LF)} = \beta_0 + \beta_1/k + \beta_2/k^2$, which has a non-null, potentially divergent value for $k \to 0$. This is in contradiction with the physical intuition that for a localized loading the low frequency limit of the velocity should be

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https://doi.org/10.1016/j.ultras.2018.07.020
Received 12 January 2018; Received in revised form 26 June 2018; Accepted 30 July 2018
Available online 31 July 2018
0041-624X/

only determined by the unmodified substrate. Therefore, both formulas are restricted to some cases which exclude the low frequency limits of localized profiles of loading.

The demonstration of Hirao et al. follows the strategy of first obtaining a perturbed solution to the wave equation. A wave field potential F is decomposed into a zeroth order and a first order term, labelled $F = F^0 + F^1$. The differential equation satisfied by F^1 has a homogeneous part identical to the original wave equation, and an inhomogeneous part involving the static stress field and F^0. By using the plane waves of the unperturbed medium, a particular solution for F^1 is constructed. Then, F is inserted into the boundary condition and a system is obtained whose determinant must vanish. This last step provides an explicit expression for the variation of velocity. The method, which is standard and, in principle, correct, is however truly cumbersome as it requires to, first, expand the inhomogeneous part of the (fourth order) differential equation satisfied by F^1, second, construct explicitly a particular solution by integrating this inhomogeneous part multiplied by products of the (four) linearly independent solutions, and finally insert the whole expansion into the boundary condition which involves several derivative operators. The expressions to deal with are thus growing considerably at each step, and it would be a true challenge to re-derive them to obtain an error-proof formula. Nevertheless, the following mistake can be identified. Hirao et al. wrote the inertial term in the wave equation $\mu V^2/V_T^2$ instead of ρV^2, to anticipate a further division by μ. This is of no consequence for the unperturbed equation, but apparently misled them to write the corresponding first order variation $2\Delta V\, V_0/V_{T,0}^2 \equiv z$ instead of $2\Delta V\, V_0/V_{T,0}^2 + \Delta\rho\, V_0^2/\mu = z - \varepsilon_{NN}\, V_0^2/V_{T,0}^2$. Indeed, in the R_i expression (see Eq. (35) in [6]), the factors of z should also be present multiplied by $-r_0\, V_0^2/V_{T,0}^2$ in the $L_i^{(2)}$ constants (see Eq. (36) in [6]), which is not the case. R_i is used to generate the particular solution F^1, so this error impacts the final formula, even though, as will be shown below, it seems to affect only slightly the predicted dispersion in the particular case considered by Hirao et al. We did not try to find further errors, nor to correct them, as we chose to follow a more compact method to obtain the perturbation formula.

Husson's demonstration is more attractive in the sense that it avoids dealing explicitly with most of the perturbed terms, and results in handling more compact expressions. By multiplying the perturbed and unperturbed fields and integrating them over a well-chosen volume, the variation of phase $\delta\Phi$ is expressed as an integral over the static stresses weighted by the unperturbed field. However, Husson's derivation is done using the wave equation expressed in the material coordinate system, *i.e.* coordinates which are deformed together with the material. As a consequence, the phase shift defined this way is bound to the deformation of the distances and must be transformed back into the space coordinate system before the variation of velocity can be deduced from it. This final step is presented as $\Delta v_R/v_R = (\Delta L/L) - \delta\Phi\, v_R/(\omega L)$, with $\Delta L/L = \bar{\varepsilon}_{11}^S$. It however happens to be a special case of a more general formula, and is valid only if the strain is uniform. So, in practice, Husson's formula is limited to uniform deformations, unless the latter transformation is replaced by the general one. Notice that Husson's article adapted its methodology from an earlier work by Husson and Kino [9] on bulk waves propagating in inhomogeneously strained media. This latter article should therefore also be considered carefully. To obtain a correct formula one strategy could be to derive this corrective term. Another one could be to re-derive the perturbation formula from the wave equation expressed in the space coordinate system, in which the velocity is measured. We have done both, although we decided to present the latter one in this article because it leads to dispersion equations that can also be solved exactly using standard numerical procedures. We will devote in the near future another article to the derivation of the general form of the corrections. Meanwhile, as a hint, we give here the general form of the relation between velocity variation and phase shift expressed in the material coordinates. By adapting Husson's demonstration to the wave equation expressed in the space coordinate system, one can define a phase shift $\delta\phi$ such as $\Delta v_R/v_R = -\delta\phi\, v_R/(\omega L)$. By transforming some terms of $\delta\phi$ into the material coordinate system, one can obtain the difference between both definitions of phase shifts:

$$\delta\phi - \delta\Phi = -\frac{\omega}{P}\int_V \frac{\mathscr{K}-\mathscr{E}}{2}\,\partial_N u_N^S\,dV - \frac{\omega}{P}\int_V \mathscr{E}_{ij}\,\partial_j u_i^S\,dV, \tag{1}$$

in which the power flow P and the densities of kinetic energy $\mathscr{K} = \frac{1}{2}\rho_0\,|\mathbf{v}_0|^2$ and elastic energy $\mathscr{E} = \mathscr{E}_{NN}$, $\mathscr{E}_{ij} = \frac{1}{2}Re\{\partial_i \mathbf{u}_0^* \cdot (\boldsymbol{\sigma}_j)_0\}$ of the dynamic field in the unperturbed medium have been defined. In the case of a Rayleigh wave in an isotropic medium, one can show that only $\mathscr{E}_{11}$ and $\mathscr{E}_{33}$ are not null, with furthermore $\int_V \mathscr{E}_{11}\,dV = L\,P/v_R$, $\int_V \mathscr{E}_{33}\,dV = 0$, $\int_V (\mathscr{K}-\mathscr{E})\,dV = 0$, *i.e.* $\delta\phi - \delta\Phi = -\bar{\varepsilon}_{11}^S(\omega L/v_R)$ if the static strain is homogeneous.

3. Basic equations

Let us consider a half space which mechanical properties are invariant in the planar (x_1, x_2) directions but may vary in the vertical x_3 direction. In its natural state, *i.e.* in absence of any mechanical deformation, the medium is described by a mass density $\rho(x_3)$, a stiffness tensor $C_{ijkl}(x_3)$ and third order elastic moduli $C_{ijklmn}(x_3)$, and its surface is isolated from any other medium. At first no assumption is made on the symmetry of the medium, although isotropy will be assumed in the next sections. A static stress $\sigma_{ij}^S(x_3) = C_{ijkl}\,\partial_k u_l^S(x_3)$ is applied and defines the initial state. Except when specified, the coordinates and derivatives refer to this state. Then, a mechanical wave of small additional amplitude is considered and defines the final state (referred to with superscript ^{f}). We shall be interested in a wave guided by the surface and propagating in the x_1 direction.

Following Pao et al. [10], in the space coordinate system defined by the initial state, the incremental displacement u_i and the difference between the final state second Piola-Kirchhoff and initial Cauchy stress tensors $T_{ij} = T_{ij}^f - \sigma_{ij}^S$ are related by the wave equation and generalized Hooke's law:

$$\partial_j[T_{ij} + \sigma_{jk}^S\,\partial_k u_i] = \rho^S\,\partial_{t^2}^2 u_i, \tag{2}$$

$$T_{ij} = C_{ijkl}^S\,\partial_k u_l, \tag{3}$$

with

$$\rho^S = \rho(1 - \partial_m u_m^S), \tag{4a}$$

$$\begin{aligned} C_{ijkl}^S = {} & C_{ijkl}(1-\partial_m u_m^S) + C_{ijklmn}\,\partial_m u_n^S \\ & + C_{mjkl}\,\partial_m u_i^S + C_{imkl}\,\partial_m u_j^S \\ & + C_{ijml}\,\partial_m u_k^S + C_{ijkm}\,\partial_m u_l^S. \end{aligned} \tag{4b}$$

If ρ^S or C_{ijkl}^S are discontinuous at some depth, then the displacement and forces are continuous through this interface. This condition must be written in the final set of coordinates, in which the wave slightly additionally modifies the space. Let us refer to this incremental deformation gradient $F_{ij}^f = \partial_j x_i^f = \delta_{ik}(\delta_{kj} + \partial_j u_k)$ and to its determinant $J^f = \det\mathbf{F}^f$. The Cauchy stress (or true stress) tensor $\boldsymbol{\sigma}^f$ is related to $\mathbf{T}^f$ through $\boldsymbol{\sigma}^f = (J^f)^{-1}\mathbf{F}^f\,\mathbf{T}^f\,{}^t\mathbf{F}^f$. An oriented surface element is transformed following $\mathbf{n}^f\,ds^f = J^f\,({}^t\mathbf{F}^f)^{-1}\,\mathbf{n}\,ds$. Using these relations, and considering $\mathbf{n} = {}^t(001)$, the elementary force through the interface expresses as:

$$\sigma_{ij}^f\,n_j^f\,ds^f = \{(\sigma_{i3}^S + T_{i3}) + (\sigma_{k3}^S + T_{k3})\,\partial_k u_i\}ds. \tag{5}$$

Remembering that σ_{ij}^S is a static stress, which therefore satisfies continuity without the presence of the incremental wave field, and neglecting the term $T_{k3}\,\partial_k u_i$ in Eq. (5), the following incremental quantity

$$\widetilde{T}_{i3} = T_{i3} + \sigma_{k3}^S\,\partial_k u_i \tag{6}$$

is continuous through the interface. At the surface, the stress-free condition of natural state expresses as $\widetilde{T}_{i3}|_{x_3=0} = 0$.

We now suppose that the wave field is harmonic in time and in the