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A note on formulas for the Rayleigh wave speed in elastic solids

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ABSTRACT

In the present paper, new analytical, numerical and approximate methods have been presented for the determination of Rayleigh wave speed in isotropic and anisotropic media. The Lagrange's method is used to provide exact expression for the roots of the secular equation for Rayleigh waves in isotropic media. Then, a simple non-iterative type quadrature method is used to numerically determine the Rayleigh wave speed in isotropic and anisotropic media. Further, an approximate method is presented to determine the velocity of Rayleigh waves. The discrete least square approximation on Chebyshev – Gauss - Lobatto nodes is suggested to transform secular equations to quadratic equations, thereby, providing improved approximations to the Rayleigh wave speed. The analysis is complemented with numerical examples.

1. Introduction

Elastic surface waves in isotropic solids, discovered by Lord Rayleigh [1], has been studied extensively in recent years, due to its wide range of applications in seismology, acoustics, geophysics, telecommunications and material science to name a few [2]. The technological applications of Rayleigh waves in electronic devices such as filters, resonators, delay lines etc. [3] has had far-reaching effects on many modern gadgets. The propagation condition for the existence of Rayleigh waves in an elastic half-space results in the secular equation for Rayleigh waves (Rayleigh equation) and its solution gives the Rayleigh wave speed [4]. Since the Green's function for many elastodynamic problems for a half-space requires the solution of the Rayleigh equation, formulas for the Rayleigh wave speed in various elastic media is of great theoretical and practical interest [5]. The Rayleigh wave equation is a cubic equation in the squared wave speed and its significance has attracted researchers to find exact, approximate analytical expression for the Rayleigh wave speed.

Rahman and Barber [6] first provided an exact expression for the roots of the Rayleigh equation in isotropic solids using the theory of cubic equations (Cardano's method). Since that time, a number of authors have sought to develop alternative expressions for the Rayleigh wave speed [7]. Nkemzi [4] provided an alternative exact expression for the Rayleigh wave speed using the theory of Cauchy Integrals, but Malischewsky [8] observed some misprints in [4] and obtained a formula for the wave speed using the advantages of computer algebra and Cardano's formula. Vinh and Ogden [9] obtained a formula solely based on the theory of cubic equations and have explained the Malischewsky formula [8]. Malischewsky Auning [10] obtained a formula for the Rayleigh wave velocity without the signum function but with an irrational term in the denominator and has shown its equivalence with the formula in [8]. Royer [11], used the root locus to provide a simple means for investigating the behaviour of the roots of the secular equation. Nkemzi [12], used a factorization technique based on the Reimann problem to derive a simple formula for the speed of Rayleigh waves. More recently, Liu and Fan [13] utilized a form of Cardano's formula (referred to in [13] as Shejun's formula) to obtained a new formula for the wave speed.

Considering anisotropic elastic solids, we note that Stoneley [14] studied the propagation of surface waves in an elastic medium with orthorhombic symmetry. The Rayleigh waves propagating in principal directions on free surfaces that are principal planes were studied. Royer and Dieulesaint [15] established that the secular equation for surface waves in orthorhombic crystals derived by Sveklo [16] could account for 16 different crystal configurations. Destrade [17] derived an explicit secular equation for surface acoustic waves in monoclinic elastic crystals using the method of first integrals. The speed of subsonic surface waves was then computed for 12 specific monoclinic crystals. Later Destrade [18]

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obtained explicit secular equations for surface waves propagating in any direction of the plane of symmetry using two different methods. Ogden and Vinh [5] obtained the secular equation for Rayleigh wave speed in an incompressible orthotropic elastic solid in a form that does not admit spurious solutions. Vinh and Ogden [19,20] obtained explicit formulas for the speed of Rayleigh waves in orthotropic compressible elastic materials using the theory of cubic equations. Each formula obtained is expressed as a continuous function of three dimensional material parameters.

In almost all the closed form expressions obtained for the Rayleigh wave velocity in isotropic and anisotropic media, the expressions are cumbersome. Recognizing the need for an approximate analytical expression for the velocity, which is simple and accurate for practical purposes, some approximate analytical expressions have been proposed in literature. One of the earliest approximate expression in this regard was proposed by Bergmann [21] which approximate the Rayleigh wave speed very well for materials with Poisson's ratio (ν) in the range [0,0.5]. Later Brekhovskikh and Godin [22], Briggs [23], Nesvijski [24] developed approximate formulas which were found to be good for positive values of ν. Malischewsky [25] proposed an approximate formula that was found to approximate the exact value of Rayleigh wave velocity for both positive and negative values of ν. However, his approach at arriving at the formula is considered to be heuristic and Rahman and Mitchelitsch [26] used the Lanczos approximation to develop an alternative approximate analytical expression for the Rayleigh wave speed in isotropic solids. This prompted Vinh and Malischewsky [27] to provide an explanation for the approximation provided in [25] using the principle of least squares. The priniciple was further used in [28] to obtain an improved approximation of Bergmann's form for the squared Rayleigh wave velocity. Royer and Clorennec [29] using a bilinear function derived a simple expression which gives an approximate value for the wave speed in isotropic solids. Vinh and Malischewsky [2] used the principle of least square to introduce an approach for finding analytical approximate formulas for the Rayleigh wave velocity in isotropic and anisotropic solids. Later the approach was used by them in [30] to obtain some improved approximations for the Rayleigh wave velocity in isotropic solids that are more accurate than the ones of the same form proposed in [20–23]. The approach was further extended in [31] to obtain improved approximation for the wave velocity in isotropic solids for Poisson ratio in the interval $[-1, 0.5]$. Li [32] used the least square approach to develop an alternative approach for obtaining approximate analytical expressions for the velocity of Rayleigh waves. More recently Spathis [7] used Pade's approximants to estimate the Rayleigh wave speed.

From the literature surveyed in the foregoing, we observe that there are strikingly four aspects involved in the studies undertaken. The primary concern being the derivation of the secular equation for Rayleigh waves in elastic media [1,2,5,14–18]. The second aspect concerns solving the secular equation to obtain an explicit analytic expression for the speed of the propagating wave [4,6,8,9,12,13,19,20]. The third is concerned with techniques that enable one to obtain approximate expressions for the Rayleigh wave speed that are as good as the exact ones [21–31]. The fourth one concerns the numerical techniques adopted to solve the secular equation to obtain the value of Rayleigh wave speed in isotropic and anisotropic elastic media. The fourth aspect is considered to be simple and as such has not been considered explicitly by previous authors. However we hope, simple techniques as expounded in this paper, would be a welcome addition to the literature.

In the present work, we consider three aspects. In the first part, analytical expressions for the roots of the secular equation of Rayleigh waves are obtained using the Lagrange's method. Secondly, we present a completely numerical approach which is a non-iterative quadrature type method for finding a simple root of the secular equation for Rayleigh waves in isotropic and anisotropic elastic solids. In the third instance, we perform a discrete least squares approximation on Chebyshev- Gauss – Lobatto (CGL) nodes to obtain approximate formulas for the Rayleigh wave velocity in elastic solids.

2. Theory

2.1. Isotropic solids

Rayleigh [1], first demonstrated the existence of Rayleigh waves and has shown that these waves are non-dispersive surface waves and the effect of these waves decrease rapidly with depth. Knowles [33], has shown that for all harmonically time-dependent free motions of the half-space, such that the plane boundary is free of traction, the displacements and stresses may be represented in terms of a scalar potential function of the surface coordinates which satisfies the two dimensional wave equation with the Rayleigh wave speed. The speed at which Rayleigh waves can propagate over the surface of an isotropic linear elastic half-space is a root of the equation.

$$\left(2-\frac{V^2}{c_2^2}\right)^2 - 4\sqrt{\left(1-\frac{V^2}{c_1^2}\right)\left(1-\frac{V^2}{c_2^2}\right)} = 0 \tag{1}$$

where $c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}}$, $c_2 = \sqrt{\frac{\mu}{\rho}}$ are the dilational and shear wave speeds of the solid and λ, μ are Lame's constants with ρ being the density. Eq. (1) can be expressed as

$$x^3 - 8x^2 + 8(3-2t)x - 16(1-t) = 0 \tag{2}$$

where $x = \frac{V^2}{c_2^2}$ and $t = \frac{c_2^2}{c_1^2} = \frac{1-2\nu}{2(1-\nu)}$, with $\nu\ (-1 \leqslant \nu \leqslant 0.5)$ being the Poisson's ratio.

The root x of Eq. (2) is found out by using the procedure outlined in Section 3.

2.2. Anisotropic solids

The modern theory of surface acoustic waves in anisotropic media owes most of its results to the pioneering work of Stroh [34,35]. A comprehensive review of the formalism, theoretical implication and schemes to compute wave speeds in anisotropic media is available in a text book by Ting [36]. For monoclinic crystals possessing atleast one plane of symmetry, secular equations which are quartic in the squared wave speed were found by Ting [37] with coefficients in terms of compliances and Destrade [17] with coefficients in terms of the stiffness.

The quartic equation for Rayleigh waves in a linear elastic semi-infinite body made of monoclinic material with a plane of symmetry at $x_3 = 0$ is given by [18].

$$d_4X^4 + d_3X^3 + d_2X^2 + d_1X - 1 = 0 \tag{3}$$

where

$$d_4 = s'_{11}(s'_{66}s'_{12}s'_{11} - s'_{66}s'^2_{11} + s'_{12}s'^2_{16} - 2s'_{16}s'_{11}s'_{26} - s'_{12}s'_{11}s'_{22} + s'^2_{16}s'_{11} + s'^3_{12} - s'_{11}s'^2_{12} + s'^2_{11}s'_{22})$$

$$d_3 = -2s'^2_{11}s'_{22} + s'^3_{11} - s'^2_{11}s'_{12} + 3s'_{66}s'^2_{11} + s'_{11}s'^2_{12} - 2s'^2_{16}s'_{11} + 4s'_{16}s'_{11}s'_{26} - 2s'_{66}s'_{12}s'_{11} + s'_{22}s'_{11}s'_{12} - s'^2_{16}s'_{12}$$

$$d_2 = -3s'_{11}s'_{16} + s'^2_{16} - 2s'_{16}s'_{26} - 3s'^2_{11} + s'_{11}s'_{22} + 2s'_{12}s'_{11} + s'_{66}s'_{12}$$

$$d_1 = 3s'_{11} - s'_{12} + s'_{66}. \tag{4}$$