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Free and forced vibrations of SC-cut quartz crystal rectangular plates with the first-order Mindlin plate equations

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ABSTRACT

Mindlin plate theory was used to provide accurate solutions to thickness-shear vibrations of plates, which have a much higher frequency than usual flexural vibrations and are the functioning modes of quartz crystal resonators. The vibration frequency solutions obtained with the Mindlin plate theory are proven being accurate along with mode shapes. In this paper, straight-crested wave solutions of free and forced vibrations of doubly rotated SC-cut of quartz crystal plates of rectangular shapes with four free edges are obtained with validated Mindlin plate equations. A procedure has been established for the calculation of dispersion relations, frequency spectra, mode shapes, and capacitance ratios of forced vibrations needed in resonator design.

1. Introduction

Mindlin plate theory was first developed and improved subsequently over half century to provide more accurate solutions to thickness-shear (TSh) vibrations of piezoelectric plates based on the power series expansion of displacements [1–7]. From the beginning, it has been widely utilized in the vibration analysis of different kinds of piezoelectric plates with the AT-cut of quartz crystal as a typical example [8–14]. For couplings of flexural and thickness-shear modes in finite crystal plates, a correction of Mindlin plate theory through inserting correction factors for Y- and AT-cut quartz crystal plates were done by Mindlin and others [15–19], effectively extending the corrected plate theory to include the thickness-twist, face-extension, and face-shear modes. The improved Mindlin plate theory is used for the analysis of crystal plates in the fundamental and overtone TSh vibrations including the consideration of effects of electrodes, temperature variation, and even the viscosity [9–14,20–23].

In the design process of piezoelectric resonators, the Mindlin plate theory is the most effective choice for the accurate analysis of vibration frequency and mode shapes. Of course, the Lee plate theory based on the trigonometric expansion is similar in nature and also offers equally accurate results of analysis as an alternative approach [24–32]. Consequently, many efforts have been made on the correction, truncation, and simplification of the Mindlin plate theory with extensive work on AT-cut quartz crystal plates with and without electrodes for resonator applications by Wang and collaborators [15–19]. The finite element method (FEM) is also implemented with the Mindlin plate equations for the numerical analysis of crystal plates, making it possible for us to examine the plate equations in detail and also deal with the two-dimensional problems more efficiently [33,34]. In this paper, the corrected Mindlin plate theory is used with the SC-cut quartz crystal plates of rectangular shapes for the analysis of thickness-shear vibrations under alternating driving voltage on electrodes. Because the SC-cut quartz crystal has enhanced material anisotropy, or more couplings of vibration modes, the analysis is generally more challenging in meeting the requirements of resonator design [12].

From our studies on the higher-order Mindlin plate equations for vibrations of overtone modes of AT-cut quartz crystal plates [9–11], a complete procedure on the correction and truncation of the Mindlin plate equations has been established and correction factors up to the fifth-order have been obtained for both analytical and FEM procedures [15–17]. In the analytical process, we have also found that the higher-order equations with correction procedure and factors are applicable to the analysis of thickness-shear vibrations of SC-cut quartz crystal plate [12–14]. With the selected vibration modes and truncated equations, we can take the advantage of the known procedure as in the analysis of AT-cut quartz crystal plates. In addition, corrected equations can also be

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implemented with the FEM based on the Mindlin plate theory for more general analysis of SC-cut quartz crystal resonators with more vibration modes in a two-dimensional approach. The analytical results from plate theories can be verified by available measurements and numerical results from a series of studies with the Lee plate theory [29–31]. In this study, we also extended applications of the Mindlin plate theory from quartz crystal plates of AT- to SC-cut for the analysis and design of resonators that have better frequency stability in applications while the design and manufacturing are considered more difficult due to complicated material processing and more coupled modes in vibrations.

2. The Mindlin plate theory

In the Mindlin plate theory, all mechanical displacements and electric potential are expanded into power series of thickness coordinate as [1–7]

$$\begin{aligned} u_j(x_1,x_2,x_3,t) &= \sum_{n=0}^{\infty} u_j^{(n)}(x_1,x_3,t)x_2^n, \quad j=1,2,3, \\ \phi(x_1,x_2,x_3,t) &= \sum_{n=0}^{\infty} \phi^{(n)}(x_1,x_3,t)x_2^n, \end{aligned} \tag{1}$$

where $u_j^{(n)},\ \phi^{(n)},\ x_j (j=1,2,3)$ and t are the nth-order displacement, nth-order electrical potential, length, thickness or width coordinate and time, respectively. This will give a set of two-dimensional equations with infinite number of modes for the analysis of plate vibrations. In practical applications, these equations have to be truncated to a set of lower-order equations for solutions. Details on the truncation and simplification of these equations can be found in earlier studies [9–11].

With Eq. (1), the nth-order strain, electric field, stress and electric displacement components in the Mindlin plate theory are defined as [1–7]

$$\begin{aligned} &S_{ij}^{(n)} = \tfrac{1}{2}\left[u_{i,j}^{(n)} + u_{j,i}^{(n)} + (n+1)(\delta_{i2}u_j^{(n+1)} + \delta_{j2}u_i^{(n+1)})\right], \\ &E_i^{(n)} = -\phi_{,i}^{(n)} - (n+1)\delta_{i2}\phi^{(n+1)}, \\ &T_{ij}^{(n)} = \sum_{m=0}^{\infty} B_{mn}\left(c_{ijkl}S_{kl}^{(m)} - e_{kij}E_k^{(m)}\right), \quad D_i^{(n)} = \sum_{m=0}^{\infty} B_{mn}\left(e_{ijk}S_{jk}^{(m)} + \varepsilon_{ij}E_j^{(m)}\right), \\ &i,j,k,l = 1,2,3, \end{aligned} \tag{2}$$

where $S_{ij}^{(n)},\ E_i^{(n)},\ T_{ij}^{(n)},\ D_i^{(n)},\ c_{ijkl},\ e_{ijk}, \varepsilon_{ij}$, and δ_{ij} are the nth-order strain, electric field, stress, electric displacement, elastic constants, piezoelectric constants, dielectric constants, and Kronecker delta, respectively, and the constant of integration in Eq. (2) is

$$B_{nm} = \begin{cases} \frac{2b^{m+n+1}}{(m+n+1)}, & m+n = \text{even}, \\ 0, & m+n = \text{odd}. \end{cases} \tag{3}$$

Consequently, the two-dimensional nth-order Mindlin plate equations of motion are [1–7]

$$\begin{aligned} &T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + F_j^{(n)} = \rho\sum_m B_{mn}\ddot{u}_j^{(m)}, \\ &D_{i,i}^{(n)} - nD_2^{(n-1)} + D^{(n)} = 0, \end{aligned} \tag{4}$$

where

$$F_j^{(n)} = [x_2^n T_{2j}]_{-b}^{b}, \quad D^{(n)} = [x_2^n D_2]_{-b}^{b}. \tag{5}$$

The mass effect of electrodes has been considered with the modified mass terms are [1–7,35]

$$\begin{aligned} &T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} = \sum_{m=0}^{\infty}\rho B_{mn}\left[1 + (m+n+1)\left(\frac{\rho'}{\rho}\frac{b'}{b} + \frac{\rho''}{\rho}\frac{b''}{b}\right)\right]\ddot{u}_j^{(m)}, \\ &D_{i,i}^{(n)} - nD_2^{(n-1)} + D^{(n)} = 0, \end{aligned} \tag{6}$$

where $\rho,\ \rho'(\rho''),\ 2b$, and $2b'(2b'')$ are the quartz crystal density, top (bottom) electrode density, quartz crystal plate thickness, and top (bottom) electrode thickness, respectively. Further consideration of the complication of electrodes can be made through the inclusion of both stiffness and mass effects of electrode layers and modification of the correction factors [16–19,33,35,36]. A simple model of SC-cut quartz crystal rectangular resonators can be well represented by a finite SC-cut rectangular quartz crystal plate with partially symmetric electrodes shown in Fig. 1, with the configuration of plate showing in dimensional parameters such as the thickness $2b$, length $2a$, and the width $2c$ and the configuration of symmetric electrodes such as the thickness $2b'$, length L, the width $2c$. The analysis of TSh vibrations of an SC-cut quartz crystal resonator is now a task of obtaining resonant vibration frequency and mode shapes under complications, and solutions, particularly displacements and electric field can be used in the calculation of electric properties such as the capacitance. The actual resonator model has more physical details and materials, but the results from a simple model shown in Fig. 1 have been widely accepted and expected for the refinement and improvement of design. Further analyses can be made by including more details in the current model like mountings and thermal effects with different methods.

3. Free vibrations of SC-cut quartz crystal rectangular plates

With equations presented in Section 2 and underlying correction, truncation, and complications, we now need to solve for displacements and electrical potential with these equations for the calculation of electrical properties of resonators. In this process, we need to obtain the dispersion relations of plates for the validation of plate equations. Then, we can study the free vibrations to examine the frequency spectra for the optimal selection of parameters of the crystal blank and electrodes. Finally, we can study the effects of driving voltage on electrodes of a resonator so complete solutions of vibrations can be obtained for the calculation of electrical properties. This has been a standard procedure practiced in the analysis of AT-cut quartz crystal resonators in the development of theoretical foundation of the analytical technique and design method [9–11,37–42]. Equations utilized in this procedure have been accepted and integrated to product development process with different methods and various implementations.

With all the zeroth- and first-order displacements and electrical potential, plate equations of motion and charge equations in Eq. (4) can be expanded as [4,5]

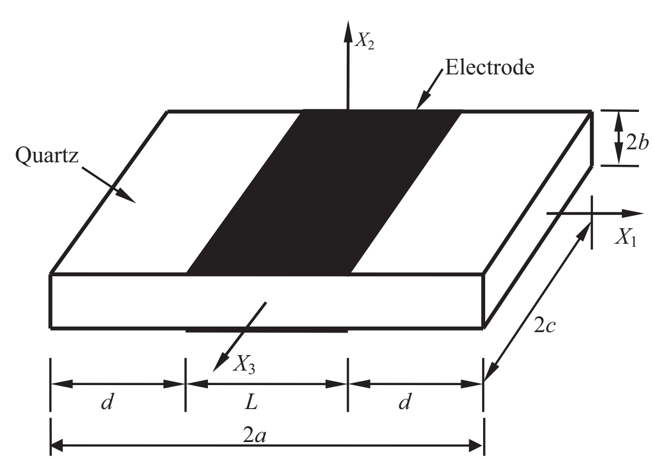

Fig. 1. An SC-cut rectangular quartz crystal plate with partially symmetric electrodes.