Available online at www.sciencedirect.com

ScienceDirect

Advances in Space Research 62 (2018) 921–934

ADVANCES IN SPACE RESEARCH
(a COSPAR publication)

www.elsevier.com/locate/asr

# Modeling approaches for precise relativistic orbits: Analytical, Lie-series, and pN approximation

Dennis Philipp[a,*], Florian Woeske[a], Liliane Biskupek[b], Eva Hackmann[a], Enrico Mai[b], Meike List[a], Claus Lämmerzahl[a], Benny Rievers[a]

[a] *ZARM, University of Bremen, 28359 Bremen, Germany*
[b] *IfE, Leibniz University, 30167 Hannover, Germany*

Received 12 February 2018; received in revised form 14 May 2018; accepted 15 May 2018

---

## Abstract

Accurate orbit modeling plays a key role in contemporary and future space missions such as GRACE and its successor GRACE-FO, GNSS, and altimetry missions. To fully exploit the technological capabilities and correctly interpret measurements, relativistic orbital effects need to be taken into account.

Within the theory of General Relativity, equations of motion for freely falling test objects, such as satellites orbiting the Earth, are given by the geodesic equation. We analyze and compare different solution methods in a spherically symmetric background, i.e. for the Schwarzschild spacetime, as a test bed. We investigate satellite orbits and use direct numerical orbit integration as well as the semi-analytical Lie-series approach. The results are compared to the exact analytical reference solution in terms of elliptic functions. For a set of exemplary orbits, we determine the respective accuracy of the different methods.

Within the post-Newtonian approximation of General Relativity, modified orbital equations are obtained by adding relativistic corrections to the Newtonian equations of motion. We analyze the accuracy of this approximation with respect to the general relativistic setting. Therefore, we solve the post-Newtonian equation of motion using the eXtended High Performance Satellite dynamics Simulator. For corresponding initial conditions, we compare orbits in the Schwarzschild spacetime to those in its post-Newtonian approximation. Moreover, we compare the magnitude of relativistic contributions to several typical perturbations of satellite orbits due to, e.g., solar radiation pressure, Earth's albedo, and atmospheric drag. This comparison is done for our test scenarios and for a real GRACE orbit to highlight the importance of relativistic effects in geodetic space missions. For the considered orbits, first-order relativistic contributions give accelerations of about 20 $\mathrm{nm/s^2}$ and are dominant in the radial direction.

*Keywords:* Relativistic geodesy; Post-Newtonian theory; Relativistic effects; Satellite orbits; Orbit propagation

---

## 1. Introduction

Contemporary and future high precision geodesy and gravimetry space missions require a precise modeling of satellite orbits. Missions such as GRACE-FO, the successor of the long-lasting gravity field recovery mission GRACE (Tapley et al., 2004), aim at nanometer accuracy in the change of the spatial distance between two spacecraft (Sheard et al., 2012; Loomis et al., 2012; Flechtner et al., 2016). At this level of accuracy, relativistic effects need to be taken into account. Therefore, precise orbit modeling and orbit propagation tools, incorporating relativistic equations of motion, are needed to consistently interpret measurements at the best possible level of accuracy. These tools usually use a numerical integration procedure and

---

* Corresponding author.
*E-mail address:* dennis.philipp@zarm.uni-bremen.de (D. Philipp).

https://doi.org/10.1016/j.asr.2018.05.020
0273-1177/

our main goal is to quantify the accuracy of this approach in two different ways outlined below.

For satellite missions and measurements, that impose a high position or orbit accuracy, accurate modeling of all physical effects acting on a satellite is essential to fully exploit measurement data. Accurate and validated force models are the basis for mission design, analysis, as well as precise orbit determination (POD) (Wu et al., 1991; Jäggi et al., 2006). POD techniques give the most accurate orbit estimates from all kind of measurement data. Nowadays, measurement methods and sensors reached an unprecedented precision such that relativistic effects must be considered, having nearly the same order of magnitude as solar radiation pressure (SRP) for conventional low Earth orbit (LEO) satellites. Exemplary missions for which highest position accuracy is required are altimetry, GNSS, geodesy, and fundamental science. For GNSS satellites, the absolute accuracy is about 5–30 cm (Steigenberger et al., 2015). For the GRACE satellites, where more measurement data are constantly available, an accuracy of less then 5 cm is achieved (Kang et al., 2003, 2006).

Within the theory of General Relativity (GR), freely falling test bodies move on timelike geodesics and the equation of motion (EOM) is given by the geodesic equation, which involves quantities derived from the spacetime metric, see, e.g., (Misner et al., 1973). For a certain class of spacetimes, which are exact solutions of Einstein's field equation with a sufficient amount of symmetries, this equation can be solved analytically, see (Forsyth, 1920; Morton, 1921; Hagihara, 1930; Darwin, 1959, 1961; Hackmann et al., 2009). Here, we choose one of these spacetimes to build trust into numerical and semi-analytical solution methods, which can be used for more complicated situations where we do not have analytical solutions at hand. We check these methods against an analytical reference solution and investigate their respective accuracy for satellite orbits around the Earth. Our approach is a first step to tackle complex but more realistic situations later on.

The post-Newtonian (pN) EOM used in advanced orbit simulation and propagation tools is an approximation of the general relativistic equation. The eXtended High Performance Satellite dynamics Simulator (XHPS) (Wöske et al., 2016) is an orbit propagation tool developed at ZARM, University of Bremen. It numerically solves the (Newtonian) EOM and is also capable of simulating the entire space environment as well as detailed satellite properties. In a second step, we therefore include pN corrections into the XHPS and compare its numerical integrator to the direct numerical integration method applied to the geodesic equation in GR, which we checked before against the analytical solution.

The purpose of this work is twofold: (I) For the application within relativistic geodesy, we aim at the comparison of different solution methods for relativistic EOM. (II) We investigate the accuracy of the first-order pN approximation of GR for satellite orbits. The pN framework also enables us to compare the magnitude of relativistic effects to various non-gravitational perturbations along satellite orbits.

We use a spherically symmetric gravitational field as a test bed. The general relativistic spacetime is then described by the Schwarzschild metric, and its first-order pN approximation involves the Newtonian gravitational potential of a point mass. This approach only includes the dominant relativistic effect on the orbits, which should, however, be sufficient for a first quantification of the accuracy of orbit simulations within the XHPS and similar tools. For the Schwarzschild spacetime, the exact solutions of the geodesic equation are well-known and given, e.g., in terms of the Weierstrass elliptic function (Hagihara, 1930). All necessary notions are introduced in Section 2, and the EOM is introduced in Section 3.

To construct orbits in the Schwarzschild spacetime, we use direct numerical integration, the semi-analytical Lie-series approach, and the analytic solution in terms of the Weierstrass elliptic function. For a pre-defined set of test orbits, the analytical solution serves as the reference to test the accuracy of the other methods. The test orbits and solution methods are introduced in Section 4, and Section 5 contains the results.

We solve the pN EOM using the XHPS that now includes relativistic corrections in the orbit propagation model at the first-order pN level. In Section 6, we access the accuracy of the pN approximation by analyzing the difference between the XHPS results and the reference orbit with corresponding initial conditions in the Schwarzschild spacetime. Finally, in Section 7 we use the XHPS and select one test orbit, for which we assume a GRACE-like satellite model, and a real GRACE orbit from 2008-04-15 to calculate the relativistic accelerations along one full orbital revolution. The result is compared to non-gravitational accelerations due to solar radiation pressure, Earth's albedo, thermal radiation pressure, and atmospheric drag.

Note that for the first-order pN approximation of the Schwarzschild spacetime, the modified Keplerian equations of motion can also be solved analytically to test the accuracy of the pN approximation. However, we use the XHPS since it allows to calculate the magnitude of various non-gravitational perturbations due to the space environment and to show that relativistic effects must be taken into account for high-precision space missions, at least at a pN level.

Table 1 shows an overview of the different solution methods for the EOM.

## 2. Geometry and notation

### 2.1. General relativistic spacetime

Within the theory of GR, the curved spacetime geometry is described by a metric $g$. In a given coordinate system, the metric components are denoted as $g_{\mu\nu}$, where we use greek indices as spacetime indices taking values $0, 1, 2, 3$.