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Topological and statistical properties of nonlinear force-free fields

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Abstract

We use our semi-analytic solution of the nonlinear force-free field equation to construct three-dimensional magnetic fields that are applicable to the solar corona and study their statistical properties for estimating the degree of braiding exhibited by these fields. We present a new formula for calculating the winding number and compare it with the formula for the crossing number. The comparison is shown for a toy model of two helices and for realistic cases of nonlinear force-free fields; conceptually the formulae are nearly the same but the resulting distributions calculated for a given topology can be different. We also calculate linkages, which are useful topological quantities that are independent measures of the contribution of magnetic braiding to the total free energy and relative helicity of the field. Finally, we derive new analytical bounds for the free energy and relative helicity for the field configurations in terms of the linking number. These bounds will be of utility in estimating the braided energy available for nano-flares or for eruptions.

Keywords: Magnetohydrodynamics (MHD); Sun: activity; Sun: corona; Sun: flares; Sun: magnetic fields

1. Introduction

The temperature of the solar corona is known to be around million degrees for decades (Grotrian, 1934; Edlén, 1943). The average density of plasma in the corona is very low $\sim 10^8\ \mathrm{cm}^{-3}$ (Aschwanden, 2004). The energy input required to compensate for the radiative and conductive losses and still maintain a million degree hot corona is estimated to be $10^7\ \mathrm{ergs\, cm^2 s^{-1}}$ for active regions and $3 \times 10^5\ \mathrm{ergs\, cm^2\, s^{-1}}$ for the quiet regions (Withbroe and Noyes, 1977; Klimchuk, 2006). The physical processes that result in the heating of the corona are not well understood, though it is believed that a key role in this is played by the magnetic fields (Schrijver and Zwaan, 2000; Golub and Pasachoff, 2010; Berger et al., 2015). The coronal heating theories can be broadly divided into two categories: direct current (DC) heating models, which are based on dissipation of magnetic stresses, and alternating current (AC) heating models which are based dissipation of waves (Ionson, 1985; Milano et al., 1997; Mandrini et al., 2000; Klimchuk, 2006). In AC heating models, it is assumed that the photospheric motion changes on a time scale faster than what the coronal loop can adjust to (e.g., by damping and dissipation of Alfvén waves), whereas in the DC heating models, it is assumed that the random photospheric motions displace the footpoints of the coronal magnetic field lines on time scales much longer than the Alfvén transit time along a coronal loop, so that the loop can adjust to the changing boundary condition in a quasi-static way. Both AC and DC models involve photospheric footpoint motions which arise from the interactions of the convective plasma flows with the magnetic flux elements (van Ballegooijen et al., 2014).

In the case of the DC heating models, the random rotations of the footpoints lead to twisting of the magnetic flux

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elements, while the random walks of these footpoints lead to their braiding (Parker, 1979; Berger and Asgari-Targhi, 2009). In order to resist the increase in complexity, the coronal magnetic field in the corona tries to adjust its topology through continuous deformations. According to Parker's magnetostatic theorem (Parker, 1972; Parker, 1988; Parker, 1994), astrophysical plasmas with high magnetic Reynolds number and a complex magnetic topology favor spontaneous generation of current sheets (resulting from sharp gradients in the magnetic field) which leads to recurrent magnetic reconnections (Kumar et al., 2016). Parker (1972, 1983), Berger (1993), and Berger and Asgari-Targhi (2009) then proposed a model which involves heating of the solar corona through nanoflares due to reconnection of braided magnetic flux elements. He further estimated the heating rate in the corona arising from the dynamical dissipation of the braided magnetic fields to be of the order of $10^7\,\mathrm{ergs\,cm^{-2}\,s^{-1}}$ (Parker, 1983) and argued it to be the principal source of heating in the active corona. The magnetic braiding can be characterized by defining a 'crossing number' which can be related to the free energy of the field (Berger, 1993). For continuous fields without distinct flux tube structures, some number N of individual field lines can be chosen within a loop, and the braiding between these lines can be quantified. Wilmot-Smith et al. (2009) presented such a semi-analytic force-free model of a pigtail braid where three magnetic field lines crossed each other six times.

However, in these studies, simple analytic configurations of magnetic fields were considered that lacked the natural complexity often observed in active regions of the Sun. Model configurations of the coronal magnetic field that are morphologically similar to those observed in the active regions, while being restricted to semi-analytic axisymmetric solutions of the linear and nonlinear force-free field (NLFFF) equation were presented in Prasad and Mangalam (2013) and Prasad et al. (2014). In Prasad et al. (2014) (hereafter PMR14), these solutions were used to simulate a library of photospheric vector magnetograms templates (depending upon the choice of parameters) that were compared with vector magnetograms observed by the spectro-polarimeter on board HINODE. This technique is complimentary to the usual approach where the magnetograms are used as a boundary condition for a numerical NLFFF extrapolation (Wiegelmann and Sakurai, 2012). The solutions are first obtained on a local spherical shell and a planar surface is placed tangential to the inner sphere that represents a Cartesian cutout of an active region (see Fig. 4 of PMR14 for more details). The orientation of the tangential plane are varied by two Euler rotations which are free parameters. The magnetic field calculated on this planar surface is then correlated with photospheric vector-magnetograms to fix the free parameters of the solutions. The radial component of magnetic field on the innermost shell is used to calculate the potential field for the volume of the shell. The three dimensional (3D) geometry of the magnetic field is used to estimate the relative helicity (Berger and Field, 1984) and the free energy (difference in magnetic helicity and energy between the NLFFF and the corresponding potential field) for the entire volume of the shell. These values are then scaled with the solid angle subtended by the magnetogram to estimate the energetics of eruptive events like solar flares. The usefulness of this method is in obtaining fast and reasonably good fits to observed vector magnetograms using semi-analytical 3D NLFFF magnetic fields.

The rest of the paper is organized as follows. In Section 2, we first present a description of the NLFFF solutions. The characterization of the amount of magnetic braiding for a toy model of two helices and for the various NLFFF solutions are presented in subsections Sections 2.1 and 2.2 using topological quantities like crossing and winding numbers and their number distributions for different modes of the NLFFF solutions are also calculated. In Section 3, we discuss linking numbers and present estimates of the free energy and relative helicity for the field configurations, and also set bounds on their magnitude. Finally, the summary and conclusions are presented in Section 4.

2. Calculation of crossing and winding for NLFFF solutions

The expression for the nonlinear force-free magnetic field in spherical geometry follows from equation (36) of PMR14 is given by

$$\mathbf{B} = \frac{-1}{r\sqrt{1-\mu^2}}\left(\frac{\sqrt{1-\mu^2}}{r}\frac{\partial\psi}{\partial\mu}\hat{\mathbf{r}} + \frac{\partial\psi}{\partial r}\hat{\boldsymbol{\theta}} - Q\hat{\boldsymbol{\phi}}\right) \tag{1}$$

where $\psi = (1-\mu^2)^{1/2}F(\mu)/r^n$, $Q = a\psi^{(n+1)/n}$, a and n are constants and $\mu = \cos\theta$. The above equation can also be obtained from Eq. (3) of Low and Lou (1990) by substituting for μ. We can then write

$$\mathbf{B} = \left(\frac{-1}{r^{n+2}}\left[(1-\mu^2)^{1/2}F'(\mu) - \frac{\mu F(\mu)}{(1-\mu^2)^{1/2}}\right],\right.$$
$$\left.\frac{n}{r^{n+2}}F, \frac{a}{r^{n+2}}(1-\mu^2)^{1/2n}F^{1+1/n}\right) \tag{2}$$

where F is obtained from

$$(1-\mu^2)F''(\mu) - 2\mu F'(\mu) + \left[n(n+1) - \frac{1}{(1-\mu^2)}\right]F(\mu)$$
$$+ a^2\frac{(n+1)}{n}F^{\frac{(n+2)}{n}}(1-\mu^2)^{\frac{1}{n}} = 0, \tag{3}$$

which has to be solved numerically as an eigenvalue problem for the variable a for a given value of n. For $n = p/q$, where p and q are integers prime to each other and $q \neq 0$, solutions exist for all odd values of p, while for even values of p, it exists only if $F(\mu) > 0$ in the domain $-1 \leqslant \mu \leqslant 1$ (PMR14). The magnetic field lines of the solutions for $n = 3$ and $m = 0 - 3$ (which correspond to different eigenvalues of a in Eq. (3)) are shown in Fig. 1. The plots are shown in a Cartesian domain (following the convention