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# Relative equilibria in quasi-homogeneous planar three body problems

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**Abstract**

In this paper we find the families of relative equilibria for the three body problem in the plane, when the interaction between the bodies is given by a quasi-homogeneous potential. The number of the relative equilibria depends on the values of the masses and on the size of the system, measured by the moment of inertia.
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*Keywords:* 3-body problems; Quasihomogeneous potentials; Relative equilibria; Central configurations

## 1. Introduction

In this paper we study a planar three body problem where the interaction between the bodies is given by a potential of the form

$$U(r) = \frac{A}{r^{\alpha}} \pm \frac{B}{r^{\beta}}, \tag{1}$$

where $r$ is the distance between the bodies, $A, B, \alpha$ and $\beta$ are positive constants. This kind of potentials are called quasi-homogeneous because they are the sum of two functions which are homogeneous and in this case with homogeneity degree $-\alpha$ and $-\beta$. Expression (1) generalizes several very well known quasi-homogeneous potentials as Birkhoff, Manev, Van der Waals, Libhoff, Schwarzschild, Lennard-Jones, the classical Newton and Coulomb and potentials that come from exact solutions of the general relativity equations (see Sthephani et al., 2003). In what follows our main purpose is to give a characterization of the special periodic solutions called relative equilibria, associated with the famous problem of central configurations (Smale, 2000). Our main contribution is to find an algebraic proof of the existence of relative equilibria in the following two situations, the attractive-attractive case and the attractive-repulsive case, which for us means that in expression (1) the components of the potential are both positive or one positive and other negative, respectively. Specifically we show that relative equilibria can correspond to arrangements of the bodies in equilateral, isosceles and scalene triangles, in function of the different values of the masses. Also we find all the possible bifurcations in the number of relative equilibria in a specific size of the system, i.e., in terms of the moment of inertia. This results generalize those presented in Corbera et al. (2004) and Arredondo and Perez-Chavela (2013).

This problem has been studied before in the specific context by several authors, some introductory aspects can be found in Corbera et al. (2004) for relative equilibria with Lennard-Jones potential in the two and three body problem with equal masses. In Arredondo and Perez-Chavela (2013) was studied relative equilibria in the three body problem with Schwarszchild potential and arbitrary values for the masses. The previous two works used numerical tools to get their conclusions. In Diacu et al. (2006) the authors provided a proof of the Moulton theorem for quasi-homogeneous potentials in general, and in Jones (2008) and Paraschiv (2012) the authors explored the nature of the central configurations and their relationship with the orbits of the bodies.

This paper is organized as follows: In Section 2 we introduce the equations of motion. In Section 3 we study the planar relative equilibria for the attractive-repulsive case,

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where we show how the number of relative equilibria depends of the size of the system and in Section 4 we extend these results to the attractive-attractive case.

## 2. Equations of motion

Let us consider systems of three bodies with masses $m_1, m_2$ and $m_3$, moving in the Euclidean plane under the influence of a quasihomoheneous type-potential and let $\mathbf{q}_i \in \mathbb{R}^2$ denote the position of the $i$-th particle in an inertial coordinate system. For this kind of systems the generalized quasihomogeneous potential takes the form

$$U(\mathbf{q}) = \sum_{i\neq j}^{3} \frac{A_{(m_i m_j)}}{r_{ij}^{\alpha}} \pm \sum_{i\neq j}^{3} \frac{B_{(m_i m_j)}}{r_{ij}^{\beta}}, \tag{2}$$

where $r_{ij} = |\mathbf{q}_i - \mathbf{q}_j|$, $\alpha$, $\beta$ are positive constants and $A_{(m_i m_j)}$, $B_{(m_i m_j)}$ are positive constants depending on the interactions between the masses $m_i$ and $m_j$, respectively with $(i, j, k)$ permuting cyclically in $(1, 2, 3)$. The equations of motion associated to the potential (2) are

$$\ddot{\mathbf{q}} = \nabla U(\mathbf{q}) \tag{3}$$

and we will assume as is usual, that the center of mass of the three particles is fixed at the origin. Henceforth, our goal will be the analysis of the *relative equilibrium*. Hence, we determine the solutions of (3) that become equilibrium points in an uniformly rotating coordinate system (see Meyer et al. (2009) for details). Relative equilibria are characterized as follows: Let $R(\omega t)$ denote the $6 \times 6$ block diagonal matrix with 3 blocks of size $2 \times 2$ corresponding to the canonical rotation in the plane. Let $\mathbf{x} \in (\mathbb{R}^6)$ be a configuration of the 3 particles, and let $\mathbf{q}(t) = R(\omega t)\mathbf{x}$ be a solution, where the constant $\omega$ is the angular velocity of the uniform rotating coordinate system. In the coordinate system $\mathbf{x}$ the equation of motion (3) becomes

$$\ddot{\mathbf{x}} + 2\omega J\dot{\mathbf{x}} = \nabla U(\mathbf{x}) + \omega^2 \mathbf{x}, \tag{4}$$

where $J$ is the usual symplectic matrix. A configuration $\mathbf{x}$ is called *central configuration* for system (3) if and only if $\mathbf{x}$ is an equilibrium point of system (4), i.e., if

$$\nabla U(\mathbf{x}) + \omega^2 \mathbf{x} = \mathbf{0}, \tag{5}$$

for some $\omega$. If $\mathbf{x}$ is a central configuration, then

$$\mathbf{q}(t) = R(\omega t)\mathbf{x} \tag{6}$$

is a relative equilibrium solution of system (3), which is also a periodic solution having period $T = 2\pi/|\omega|$. Therefore, when a central configuration is obtained, one has also the corresponding relative equilibria, and that is why central configurations and relative equilibria are equivalent concepts.

Is it worth to mention, for the sake of clarity, that Eq. (5) for a central configuration $\mathbf{q} = \mathbf{x}$, says that a central configuration in the space $\mathbf{q}$ is a particle configuration for which; the position $\mathbf{q}$ and the acceleration $\ddot{\mathbf{q}}$ vectors of each particle are proportional, with the same constant of proportionality $\omega^2$.

## 3. Attractive-repulsive case

In this section we consider that the interaction between the bodies correspond to a quasi-homogeneous potential, with one attractive component and other repulsive. Thus expression (2) can be written as

$$U(\mathbf{q}) = \sum_{i\neq j}^{3} \frac{A_{(m_i m_j)}}{r_{ij}^{\alpha}} - \sum_{i\neq j}^{3} \frac{B_{(m_i m_j)}}{r_{ij}^{\beta}}. \tag{7}$$

Since in this case central configurations are not invariant under homotheties, it is natural to think that their number depends on the size of the system measured by the moment of inertia $I$, which can be written in terms of the mutual distances as

$$I = \frac{1}{M}\left(m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2\right), \tag{8}$$

where $M = m_1 + m_2 + m_3$. For this case, the following theorem presents the main result.

**Theorem 1.** *Consider the planar* 3-*body problem, where the mutual interaction between the particles is given on the form presented in* (7), *then,*

1. *If the three masses are equal, the relative equilibria can be equilateral or isosceles triangles. The number of relative equilibria depend on the moment of inertia, and there are four bifurcation values for I.*
2. *If two masses are equal, for any value of the constants $A_{(m_i m_j)}$ and $B_{(m_i m_j)}$ the relative equilibria can be isosceles triangles, and can be equilateral and scalene triangles for special values of this constants.*
3. *If the three masses are different, for any value of the constants $A_{(m_i m_j)}$ and $B_{(m_i m_j)}$ the relative equilibria can be scalene triangles, and can be equilateral and isosceles triangles for special values of the constants.*

**Proof.** First all, let us remember the next Lemma whose proof appears in Corbera et al. (2004). □

**Lemma 1.** *Let* $u = f(x), x = (x_1, \dots x_n)$, $x_1 = g_1(y), \dots, x_n = g_n(y)$ *with* $y = (y_1, \dots y_m)$, $m \geqslant n$. *If* $\mathrm{rank}(A) = n$ *where*

$$A = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \\ \vdots & \ddots & \\ \frac{\partial x_1}{\partial y_m} & & \frac{\partial x_n}{\partial y_m} \end{pmatrix},$$

*then* $\nabla f(x) = 0$ *if and only if* $\nabla u(y) = 0$.

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