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# On the convergence of the theory of figures

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ABSTRACT

The so-called theory of figures (TOF) uses potential theory to solve for the structure of highly distorted rotating liquid planets in hydrostatic equilibrium. An apparently divergent expansion for the gravitational potential plays a fundamental role in the traditional TOF. This questionable expansion, when integrated, leads to the standard geophysical expansion of the external gravitational potential on spherical-harmonics (via the usual *J*-coefficients). We show that this expansion is convergent and exact on the planet's level surfaces, provided that rotational distortion does not exceed a critical value. We examine the general properties of the Maclaurin multipole expansion and discuss conditions for its convergence on the surface of both single and nested-concentric Maclaurin spheroids.


## 1. Introduction

Hubbard (2012) introduced a numerical scheme for iteratively obtaining the external multipole gravity harmonics of a rotating constant-density Maclaurin spheroid. He compared numerical results for specified rotation parameters with Maclaurin's exact solution and found agreement to high precision, for rotation parameters typical of giant planets. Kong et al. (2013) showed that the Hubbard scheme diverges when the spheroid's rotational distortion parameter is not sufficiently small, and demonstrated an alternate expansion that converges for all values of the parameter. Neither paper fully clarified the exact domain where the two approaches might overlap, nor the magnitude of discrepancies (if any) between the approaches. A relevant issue is the validity of the so-called Laplace expansion, as discussed in Zharkov and Trubitsyn (1978). The latter reference includes a systematic exposition of the perturbation approach to TOF.

The purpose of the present paper is to reconcile, analytically and numerically, the different approaches of Hubbard (2012), Kong et al. (2013) and Zharkov and Trubitsyn (1978), as well as the newer nonperturbative method of Hubbard (2013), and to clearly delineate each method's range of validity and expected precision for modeling planetary external gravity fields.

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## 2. Power-series expansions of the gravitational potential

### 2.1. The laplace expansion

The gravitational potential $V$ at a point at vector coordinate $\mathbf{r}$ due to a distribution of mass density $\rho$ is given by

$$V(\mathbf{r}) = G\int d^3r'\rho(\mathbf{r}')/|\mathbf{r}-\mathbf{r}'| \tag{1}$$

where $G$ is the gravitational constant and the integral is taken over all space where $\rho \neq 0$. For a body rotating at rate $\omega$, the total potential in the fluid's frame is $U = V + Q$, where the rotational potential $Q$ is proportional to $\omega^2$.

Consider the simplified problem of the Maclaurin spheroid with $\rho = \text{const}$. Fig. 1 shows a cross-sectional view of its surface, enclosed by reference spheres of radius $a$ (the spheroid's equatorial radius), and radius $b$ (the spheroid's polar radius). The spheroid's surface radius $r(\mu)$ is an ellipsoid of revolution given by

$$r^2 = \frac{a^2}{1+\ell^2\mu^2}, \tag{2}$$

where $\mu$ is the cosine of the angle from the rotation axis, and

$$\ell^2 = \frac{a^2}{b^2} - 1. \tag{3}$$

As discussed by Zharkov and Trubitsyn (1978), Laplace posed the problem of finding approximations to $V$ by expanding the factor $|\mathbf{r}-\mathbf{r}'|^{-1}$ under the integral in Eq. (1). To calculate $V$ on the spheroid's surface, Laplace expanded $|\mathbf{r}-\mathbf{r}'|^{-1}$ in powers of $r'/r$,

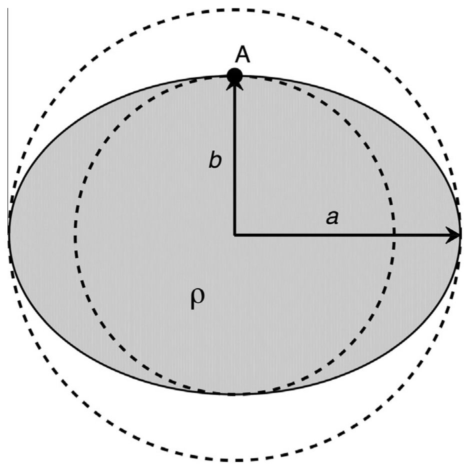


**Fig. 1.** The solid curve shows the ellipsoidal surface of a Maclaurin spheroid, while the outer dashed curve is a reference sphere of radius $a$ which osculates the spheroid's equator. The inner dashed curve is a reference sphere of radius $b$ which osculates the spheroid's pole. "A" is an audit point situated at the pole. The spheroid shown here has $\ell^2 = 1$.

such that the potential on and exterior to the spheroid's surface becomes a sum over multipole moments $J_{2n}$ of the interior mass distribution, with each moment multiplied by an appropriate Legendre polynomial $P_{2n}(\mu)$ and factor $(a/r)^{2n+1}$. The expansion converges at all points exterior to the outer dashed sphere in Fig. 1. However, between the two dashed spheres, which of course includes all points on the spheroid's surface, there will be a contribution to the integrand from regions with $|r'/r| \geqslant 1$, where the expansion in powers of $r'/r$ diverges.

Nevertheless, Zharkov and Trubitsyn (1978) conclude that "… use of a divergent Legendre series is quite valid, because the series becomes unconditionally convergent after integration, and corresponds to expansion in powers of $\omega^2$." As we show below, this claim is correct, but if and only if the rotational distortion is sufficiently small.

### 2.2. Expansion in powers of a small parameter

The general TOF of Zharkov and Trubitsyn (1978), which makes use of the (questionable) Laplace expansion, yields a hierarchy of expressions for the external gravitational potential. One may define a small dimensionless parameter, whose leading term is proportional to $\omega^2$, to measure the amplitude of the rotational distortion and corresponding planetary response. Commonly-used small parameters include

$$q = \frac{\omega^2 a^3}{GM}, \tag{4}$$

where $M$ is the planetary mass, and

$$m = \frac{3\omega^2}{4\pi G\rho}, \tag{5}$$

where $\rho$ is the planet's mean density. To lowest order in $\omega^2$, the small parameters $m$ and $q$ are equivalent.

For the particular case of the Maclaurin spheroid ($\rho = \text{const.}$), the parameter $\ell^2$ can also be used as a small parameter, and one has the exact relation (Lamb, 1993)

$$m = \frac{3}{2\ell^3}\left[(3+\ell^2)\tan^{-1}\ell - 3\ell\right]. \tag{6}$$

Zharkov and Trubitsyn (1978) showed that the external gravitational potential of a rotating planet in hydrostatic equilibrium can be expressed as a *double power-series* expansion in the small parameter. Each multipole term in the expansion for $V$ has a coefficient whose further expansion takes the form

$$J_{2n} = m^n \sum_{t=0}^{\infty} \Lambda_{2n}^{(t)} m^t, \tag{7}$$

where the dimensionless response coefficients $\Lambda_{2n}^{(t)}$ are obtained from the solution of a hierarchy of integrodifferential equations. For practical calculations in the Zharkov–Trubitsyn TOF, the expansion in Eq. (7) must be truncated at a prescribed limiting order in $m$ (the most complex model considered by Zharkov and Trubitsyn (1978) carries the calculation to order $m^5$). The total potential $V$ is thus a double power-series expansion in $m$, since $V$ is a weighted sum over the $J_{2n}$.

For the special case of the Maclaurin spheroid, Hubbard (2012) showed that the double power-series expansion for the external gravitational potential becomes a single expansion over the $J_{2n}$, since each of these terms can be written in the closed (non-perturbative) form

$$J_{2n} = \frac{3(-1)^{1+n}}{(2n+1)(2n+3)}\left(\frac{\ell^2}{1+\ell^2}\right)^n, \tag{8}$$

Note that Eq. (8) can be rewritten in a perturbative (double power-series) form equivalent to Eq. (7) by expanding the expression for $J_{2n}$ in powers of $\ell^2$, and then using Eq. (6) to expand $\ell^2$ in powers of $m$.

### 2.3. Surface potential of the Maclaurin spheroid

In exact closed form, the surface potential of a Maclaurin spheroid is given by (Zharkov et al., 1971; Lamb, 1993)

$$U_{\text{surface}} = \frac{3GM}{2a}\sqrt{1+\ell^2}\left[\left(\frac{1}{\ell}+\frac{1}{\ell^3}\right)\tan^{-1}\ell - \frac{1}{\ell^2}\right]. \tag{9}$$

Let us now compare this exact result with the surface potential obtained from a traditional geophysical expansion of the external gravitational potential (Zharkov and Trubitsyn, 1978):

$$U_{\text{surface,geophys}} = \frac{GM}{a}\left[\xi^{-1} - \sum_{k=1}^{\infty} J_{2k}\xi^{-(2k+1)}P_{2k}(\mu) + \frac{q}{2}\xi^2(1-\mu^2)\right], \tag{10}$$

where $\xi = r/a$, and the final term corresponds to the rotational potential $Q$.

Substituting $\xi(\mu)$ from Eq. (2) into Eq. (10), and expanding each $\xi^{-(2k+1)}$ in powers of $\ell^2$ and $\mu^2$, we obtain an expansion for $U_{\text{surface,geophys}}$ as a bivariate polynomial in $\ell^2$ and $\mu^2$. Since $U$ must be constant on the spheroid surface, the coefficients of all terms in $\mu^2$ must equate to zero. This proves to be the case, and the surviving $\mu$-independent terms give

$$U_{\text{surface,geophys}} = \frac{GM}{a}\left[1 + \frac{3}{10}\ell^2 - \frac{39}{280}\ell^4 + \frac{139}{1680}\ell^6 - \frac{2749}{49280}\ell^8 + \cdots\right]. \tag{11}$$

Next, we seek an expansion of Eq. (9) for $U_{\text{surface}}$ as a power series in $\ell^2$. For this purpose, we must use the Taylor series

$$\tan^{-1}\ell = \ell - \frac{1}{3}\ell^3 + \frac{1}{5}\ell^5 - \frac{1}{7}\ell^7 + \cdots. \tag{12}$$

The resulting power series for $U_{\text{surface}}$ is identical to Eq. (11). However, a Taylor series expansion of Eq. (9) in $\ell$ only converges for $\ell < 1$. Thus we confirm the result of Kong et al. (2013), that the geophysical expansion for the surface potential of the Maclaurin spheroid is convergent if and only if $\ell < 1$. Fig. 1 shows the shape of a Maclaurin spheroid for the critical case $\ell = 1$. This case is critical in a mathematical sense only: it corresponds to the failure of the Taylor series expansion of Eq. (9) and hence the failure of the equivalent geophysical expansion of Eq. (10) to converge