New Astronomy 43 (2016) 22–25

Contents lists available at ScienceDirect

New Astronomy

journal homepage: www.elsevier.com/locate/newast

Effects of zonal harmonics on the out-of-plane equilibrium points in the generalized Robe's circular restricted three-body problem

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HIGHLIGHTS

- We included the oblateness coefficient up to the second zonal harmonics.
- We examined the existence of the out-of-plane equilibrium points.
- We investigated the effects of the zonal harmonics on the out-of-plane equilibrium points.
- We investigated the effects of the zonal harmonics on stability of these points.

ARTICLE INFO

Article history:
Received 8 August 2014
Revised 18 May 2015
Accepted 13 July 2015
Available online 21 July 2015

Keywords:
Zonal harmonics
Out-of-plane equilibrium points
Linear stability

ABSTRACT

This article examines the effects of the zonal harmonics on the out-of-plane equilibrium points of Robe's circular restricted three-body problem when the hydrostatic equilibrium shape of the first primary is an oblate spheroid, the shape of the second primary is an oblate spheroid with oblateness coefficients up to the second zonal harmonic, and the full buoyancy of the fluid is considered. It is observed that the size of the oblateness and the zonal harmonics affect the positions of the out-of-plane equilibrium points L_6 *and* L_7. It is also observed that these points within the possible region of motion are unstable.

1. Introduction

The out-of-plane points have no analogy in the classical restricted three-body problem. However, these points were first examined in the photogravitational restricted three-body problem by Radzievskii (1950). Afterward, many researchers, for example, Douskos and Markerllos (2006) and Singh and Leke (2010), worked on the out-of-plane points and all presented interesting results.

A new kind of restricted three-body problem was formulated by Robe (1977), in which the first primary of mass m_1 was a rigid spherical shell, filled with a homogeneous, incompressible fluid of density ρ_1, the second primary was a point mass m_2 outside the shell; and the third primary of mass m_3 was a small solid sphere of density ρ_3 moving inside the shell. He assumed that the mass and radius of m_3 are infinitesimal, and showed the center of the first primary as an equilibrium point and examined its linear stability in two cases. In the first case, the orbit of m_2 around m_1 is circular, and in the second case, it is elliptic; however, the shell is empty (*no fluid inside it*), or the densities ρ_1 and ρ_3 are equal (Fig. 1).

During the evaluation of buoyancy force, Robe (1977) assumed that the pressure field of the fluid with density ρ_1 has a spherical symmetry around the center of the shell, because of its own gravitational field. The remaining two components are attraction of m_1, arising from the centrifugal force, and attraction of m_2. All these components of the pressure field are included in the study of Robe's problem by Plastino and Plastino (1995). However, they assumed the hydrostatic equilibrium shape of the first primary as a Roche ellipsoid. They also established conditions for the existence of libration points and their stability.

Singh and Sandah (2012) investigated the equilibrium points and their stability in the Robe's circular restricted three-body problem when the hydrostatic equilibrium shapes of the first and second primaries are similar, that is, an oblate spheroid. Singh and Mohammed (2013) extended Hallan and Mangang's (2007) work by assuming the second primary as a triaxial rigid body. Their most interesting and distinguishable results were the existence of elliptical and off-plane points, and their stability. Singh and Sandah (2012) considered only the first zonal harmonic in deriving the expression for the oblateness of the second primary.

The aim of this article is to include the aforementioned oblateness coefficients up to the second zonal harmonic and to investigate the out-of-plane equilibrium points, and their linear stability.

This article is divided into five sections, including this section of introduction. The next section describes the equations of motion. The positions of the out-of-plane points are studied in Section 3,

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http://dx.doi.org/10.1016/j.newast.2015.07.003
1384-1076/

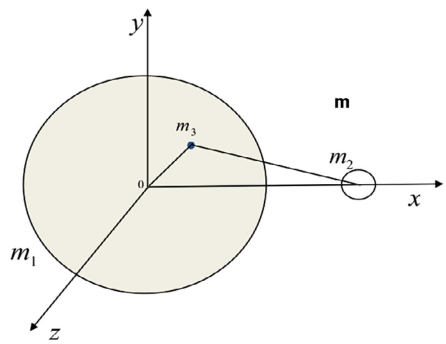

Fig. 1. Robe's circular restricted three-body problem.
The first primary of mass m_1 is a rigid spherical shell, filled with a homogeneous, incompressible fluid of density ρ_1; the second one is a point mass m_2 outside the shell; and the third body of mass m_3 is a small solid sphere of density ρ_3 moving inside the shell.

and their stability and conclusion are provided in Sections 4 and 5, respectively.

2. Equations of motion

We assume that the first primary of mass m_1 is a fluid of density ρ_1 with an oblate spheroid shape, and the second primary of m_2 is also an oblate spheroid with oblateness coefficients up to the second zonal harmonics. The mass m_2 moves in a circular orbit around m_1, and the infinitesimal mass m_3 moves inside the first primary. Considering a uniformly synodic coordinate system $0x_1x_2x_3$, where the center of m_1 is the origin, $0x_1$ points toward m_2, and $0x_1x_2$, being the orbital plane of m_2, coincides with the equatorial plane of m_1. Then, the equations of motion of the infinitesimal mass of density ρ_3 in this coordinate system, as provided by Singh and Sandah (2012) and Singh and Omale (2014), are given as:

$$\ddot{x}_1 - 2n\dot{x}_2 = \frac{\partial U}{\partial x_1}, \quad \ddot{x}_2 + 2n\dot{x}_1 = \frac{\partial U}{\partial x_2}, \quad \ddot{x}_3 = \frac{\partial U}{\partial x_3}, \qquad (2.1)$$

where,

$$U = V + \frac{n^2}{2}\left[(x_1 - \mu R)^2 + x_2^2\right],$$

$$V = B + B' - \frac{\rho_1}{\rho_3}\left[B + B' + \frac{n^2}{2}\left\{(x_1 - \mu R)^2 + x_2^2\right\}\right],$$

$$B = \pi G\rho_1\left[I - A_1x_1^2 - A_1x_2^2 - A_2x_3^2\right]$$

$$B' = \frac{Gm_2}{\left[(R-x_1)^2 + x_2^2 + x_3^2\right]^{1/2}} + \frac{Gm_2\alpha_2}{2\left[(R-x_1)^2 + x_2^2 + x_3^2\right]^{3/2}} - \frac{3Gm_2\alpha_2x_3^2}{2\left[(R-x_1)^2 + x_2^2 + x_3^2\right]^{5/2}} + \frac{3Gm_2\alpha_3x_3^2}{8\left[(R-x_1)^2 + x_2^2 + x_3^2\right]^{7/2}},$$

$$I = 2a_1^2A_1 + a_2^2A_2, \quad A_1 = a_1^2a_2\int_0^\infty \frac{du}{\Delta\left(a_1^2 + u\right)},$$

$$A_2 = a_1^2a_2\int_0^\infty \frac{du}{\Delta\left(a_2^2 + u\right)},$$

$$\Delta^2 = \left(a_1^2 + u\right)^2\left(a_2^2 + u\right),$$

$$n^2 = \frac{G(m_1 + m_2)}{R^2}\left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3\right),$$

$$\alpha_1 = \frac{a_1^2 - a_2^2}{5R^2}, \quad \alpha_2 = J_2R_2^2, \quad \alpha_3 = J_4R_2^2,$$

Here, the mean radius of m_2 *is* R_2, and J_2 and J_4 are the zonal harmonic coefficients that characterize the size of the nonspherical components of the potential. The potential due to the combined forces acting upon the infinitesimal mass is V, B denotes the potential due to the fluid mass of the first primary, B' denotes the potential due to the second primary, R is the distance between the primaries, G is the gravitational constant, n is the mean motion, a_1 and a_2 are the equatorial and polar radii of the first primary, I stands for the polar moment of inertia, and A_i $(i = 1,\ 2)$ are the index symbols.

We assume that the sum of the masses and the distance between the primaries is unity. Thus, we let $m_2 = \mu$, $0 < \mu = \frac{m_2}{(m_1 + m_2)} < 1$. The unit of time is also selected such that $G = 1$. On the basis of these assumptions, the equations of motion in Eq. (2.1) are given as:

$$\ddot{x}_1 - 2n\dot{x}_2 = \frac{\partial U}{\partial x_1}, \quad \ddot{x}_2 + 2n\dot{x}_1 = \frac{\partial U}{\partial x_2}, \quad \ddot{x}_3 = \frac{\partial U}{\partial x_3}, \qquad (2.2)$$

where

$$U = D\left[\pi\rho_1\left\{I - A_1\left(x_1^2 + x_2^2\right) - A_2x_3^2\right\} + \frac{\mu}{2r}\left(2 + \frac{\alpha_2}{r^2} - \frac{3\alpha_2x_3^2}{r^4} + \frac{3\alpha_3x_3^2}{4r^6}\right) + \frac{n^2}{2}\left\{(x_1 - \mu)^2 + x_2^2\right\}\right]$$

$$\mathrm{r} = \left[(1 - x_1)^2 + x_2^2 + x_3^2\right]^{1/2}, \ n^2 = 1 + \frac{3\alpha_1}{2} + \frac{3\alpha_2}{2} - \frac{15\alpha_3}{8},$$

$$0 < \mu < 1, \ \alpha_1, \alpha_2, \ \alpha_3 \ll 1, \ D = 1 - \frac{\rho_1}{\rho_3}.$$

Eq. (2.2) provides the equations of motion of the infinitesimal mass under the influences of the full buoyancy force of the fluid, oblateness, and gravitational forces of the primary bodies. They differ from those of Hallan and Mangang (2007) on the presence of the first two zonal harmonic coefficients of the second primary, and from those of Singh and Sandah (2012) on the appearance of their second zonal harmonic coefficient.

3. Positions of out-of-plane equilibrium points

These points are the solutions of the following equations:

$$U_{x_1} = U_{x_2} = U_{x_3} = 0, \ with\ x_2 = 0 \text{ and } D \neq 0, \qquad (3.1)$$

that is

$$x_1\left[-2\pi\rho_1A_1 - \frac{\mu}{\left[(1 - x_1)^2 + x_3^2\right]^{3/2}} - \frac{3\mu\alpha_2}{2\left[(1 - x_1)^2 + x_3^2\right]^{5/2}} + \frac{15\mu\alpha_2x_3^2}{2\left[(1 - x_1)^2 + x_3^2\right]^{7/2}} - \frac{21\mu\alpha_3x_3^2}{8\left[(1 - x_1)^2 + x_3^2\right]^{9/2}} + n^2\right] + \frac{\mu}{\left[(1 - x_1)^2 + x_3^2\right]^{3/2}} + \frac{3\mu\alpha_2}{2\left[(1 - x_1)^2 + x_3^2\right]^{5/2}} - \frac{15\mu\alpha_2x_3^2}{2\left[(1 - x_1)^2 + x_3^2\right]^{7/2}} + \frac{21\mu\alpha_3x_3^2}{8\left[(1 - x_1)^2 + x_3^2\right]^{9/2}} - n^2\mu = 0 \qquad (3.2)$$

and

$$x_3\left[-2\pi\rho_1A_1 - \frac{\mu}{\left[(1 - x_1)^2 + x_3^2\right]^{3/2}} - \frac{9\mu\alpha_2}{2\left[(1 - x_1)^2 + x_3^2\right]^{5/2}} + \frac{15\mu\alpha_2x_3^2}{2\left[(1 - x_1)^2 + x_3^2\right]^{7/2}} + \frac{6\mu\alpha_3}{8\left[(1 - x_1)^2 + x_3^2\right]^{7/2}} - \frac{21\mu\alpha_3x_3^2}{8\left[(1 - x_1)^2 + x_3^2\right]^{9/2}}\right] = 0. \qquad (3.3)$$