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Critical rotation of general-relativistic polytropic models simulating neutron stars: A post-Newtonian hybrid approximative scheme

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HIGHLIGHTS

- A "hybrid approximative scheme" (HAS) is developed and discussed in detail.
- Computations by HAS focus on maximum-mass critically rotating polytropic models.
- Comparisons show that HAS derives numerical results with satisfactory accuracy.

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ABSTRACT

We develop a "hybrid approximative scheme" in the framework of the post-Newtonian approximation for computing general-relativistic polytropic models simulating neutron stars in critical rigid rotation. We treat the differential equations governing such a model as a "complex initial value problem", and we solve it by using the so-called "complex-plane strategy". We incorporate into the computations the complete solution for the relativistic effects, this issue representing a significant improvement with regard to the classical post-Newtonian approximation, as verified by extended comparisons of the numerical results.

1. Introduction

The original contributions to the study of rapidly rotating neutron stars in the framework of the "post-Newtonian approximation" (PNA) are due to Chandrasekhar (1965), Krefetz (1967), and Fahlman and Anand (1971). The problem of fast rigid rotation of neutron stars in hydrostatic equilibrium is treated in Fahlman and Anand (1971) by considering the relativistic and rotational effects acting on a nonrotating Newtonian configuration obeying the polytropic "equation of state" (EOS, EOSs). However, there are certain reasons leading the PNA of first-order in the gravitation parameter σ to failure when $\sigma \geqslant 0.01$. A discussion on this matter can be found in Tooper (1965) (Appendix). A further discussion ((Fahlman and Anand, 1971), Section 5) verifies the negative conclusions of Tooper (1965) and focuses on the imposed limitations when applying this PNA's scheme to several astrophysical objects, since, unfortunately, values of interest lie in the vicinity of $\sigma \simeq 0.1$.

In a recent study (Geroyannis and Karageorgopoulos, 2014), we revisit the problem by assuming the relativistic and rotational effects as decoupled perturbations, and by applying to PNA the so-called "complex plane strategy" (CPS). This method consists in solving all differential equations involved in the PNA's computational scheme in the complex plane. Numerical integrations are resolved by the Fortran code DCRKF54 (Geroyannis and Valvi, 2012), which is a Runge–Kutta–Fehlberg code of fourth and fifth order, modified so that to integrate "initial value problems" (IVP, IVPs) established on systems of first-order "ordinary differential equations" (ODE, ODEs) of complex-valued functions in one complex variable along prescribed complex paths.

As discussed in Geroyannis and Karageorgopoulos (2014) (Section 5.2), CPS could proceed independently of the particular perturbation approach used. For instance, CPS could be applied to a PNA's scheme of up to second order in σ, as developed in Chandrasekhar and Nutku (1969). But, most interesting, CPS could cooperate with a "hybrid approximative scheme" (HAS) of PNA ((Geroyannis and Karageorgopoulos, 2014), Section 5.2), in which the complete solution of the relativistic distortion, as developed in Tooper (1965), is involved. In this study, we extend the

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numerical experiments started in Geroyannis and Karageorgopoulos (2014) (Section 5.2), by applying HAS to general-relativistic polytropic models of critically rotating neutron stars with σ up to ~ 0.8 (holding for $n = 0.5$).

We do not intend to repeat here extended parts of Geroyannis and Karageorgopoulos (2014), except for certain significant issues. For clarity and convenience, we use the same conventions, definitions, and symbols with those in Geroyannis and Karageorgopoulos (2014).

2. The hybrid approximative scheme

2.1. Preliminaries

In this study, we assume that the pressure p and the rest-mass density ρ obey the polytropic "equation of state" (EOS)

$$p = K\,\rho^{\Gamma} = K\,\rho^{1+(1/n)}, \tag{1}$$

where K is the polytropic constant, Γ the adiabatic index defined by $\Gamma = 1 + (1/n), n$ the polytropic index, and the normalization equations for the rest-mass density ρ and the coordinate r are defined by

$$\rho = \rho_c\,\Theta^n, \quad r = \left[\frac{(n+1)p_c}{4\pi G\rho_c^2}\right]^{1/2}\xi = \left[\frac{(n+1)K\rho_c^{(1/n)}}{4\pi G\rho_c}\right]^{1/2}\xi = \alpha\xi, \tag{2}$$

where ρ_c is the central density, $\Theta(\xi, \mu)$ (with $\mu = \cos(\vartheta)$) the Lane–Emden function, p_c the central pressure, and G the gravitation constant. The central density ρ_c is chosen to be the density unit in the so-called "classical polytropic units" (cpu), and the model parameter α is chosen to be the length unit in cpu; accordingly, Θ^n is the cpu measure of the rest-mass density ρ and ξ the cpu measure of the coordinate r.

The "rotation parameter" υ, representing the effects of rotation, and the "gravitation parameter" σ (also called "relativity parameter"), representing the post-Newtonian effects of gravitation, are then defined by (Geroyannis and Karageorgopoulos, 2014, Eqs. (7a) and (7b), respectively)

$$\upsilon = \frac{\Omega^2}{2\pi G\rho_c}, \qquad \sigma = \frac{1}{c^2}\,\frac{p_c}{\rho_c}. \tag{3}$$

In the framework of PNA, the function $\Theta(\xi, \mu)$ can be expressed as (Geroyannis and Karageorgopoulos, 2014, Eq. 9)

$$\begin{aligned}\Theta(\xi,\mu) &= \sum_{i=0,2}^{4} P_i(\mu)\,\Theta_i(\xi) \\ &= \alpha_0\,\theta_{00}(\xi)\,P_0(\mu) + \alpha_1[\theta_{10}(\xi)\,P_0(\mu) + A_{12}\theta_{12}(\xi)P_2(\mu)] \\ &\quad + \alpha_2\{\theta_{20}(\xi)\,P_0(\mu) + [\theta_{22}(\xi) + A_{22}\theta_{12}(\xi)]P_2(\mu) \\ &\quad + [\theta_{24}(\xi) + A_{24}\theta_{14}(\xi)]P_4(\mu)\} + \alpha_3\,\theta_{30}(\xi)\,P_0(\mu),\end{aligned} \tag{4}$$

where α_i are the perturbation parameters (Fahlman and Anand, 1971, Eq. (24)): $\alpha_0 = 1, \alpha_1 = \upsilon, \alpha_2 = \upsilon^2$, and $\alpha_3 = \sigma$. The functions θ_{ij} are involved in the differential equations (Geroyannis and Karageorgopoulos, 2014, Eq. (12))

$$\frac{d^2\theta_{ij}}{d\xi^2} + \frac{2}{\xi}\,\frac{\theta_{ij}}{d\xi} - \frac{j(j+1)}{\xi^2}\,\theta_{ij} = S_{ij} \tag{5}$$

with $i = 0, 1, 2, 3$, and $j = 0, 2, 4$, solved in view of the initial conditions (26) of Geroyannis and Karageorgopoulos (2014). The parameters A_{ij} (Geroyannis and Karageorgopoulos, 2014, Eqs. (24)–(25)) multiply properly the homogeneous solutions of θ_{ij} (Fahlman and Anand, 1971, Eqs. (42) and (43)), so that the boundary conditions (16) of Geroyannis and Karageorgopoulos (2014) be satisfied. The functions S_{ij} are given by Eq. (13) of Geroyannis and Karageorgopoulos (2014).

2.2. The numerical method

We now consider HAS as a computational scheme applied on PNA of Geroyannis and Karageorgopoulos (2014), in which the relativistic distortion participates with its complete solution, as it has been developed and computed in Tooper (1965). By substituting the complete solution Θ_σ for the relativistic effects in the place of the sum $\alpha_0\,\theta_{00}(\xi) + \alpha_3\,\theta_{30}(\xi)$ (Geroyannis and Karageorgopoulos, 2014, Eq. (57)), we obtain the form

$$\begin{aligned}\Theta(\xi,\mu) &= \Theta_\sigma\,P_0(\mu) + \alpha_1[\theta_{10}(\xi)\,P_0(\mu) + A_{12}\theta_{12}(\xi)P_2(\mu)] \\ &\quad + \alpha_2\{\theta_{20}(\xi)\,P_0(\mu) + [\theta_{22}(\xi) + A_{22}\theta_{12}(\xi)]P_2(\mu) \\ &\quad + [\theta_{24}(\xi) + A_{24}\theta_{14}(\xi)]P_4(\mu)\}.\end{aligned} \tag{6}$$

To compute the function Θ_σ, we use the Oppenheimer–Volkoff equations of hydrostatic equilibrium (cf. Tooper, 1965, Eqs. (19) and (20)),

$$\frac{d\Theta_\sigma}{d\xi} = -\frac{1}{\xi^2}\left(\Upsilon_\sigma + \sigma\,\xi^3\,\Theta_\sigma^{n+1}\right)\frac{[1 + (n+1)\,\sigma\,\Theta_\sigma]}{1 - 2\,\sigma\,(n+1)\,\Upsilon_\sigma/\xi}, \tag{7}$$

$$\Upsilon'_\sigma = \xi^2\,\Theta_\sigma^n(1 + \sigma\,n\,\Theta_\sigma), \tag{8}$$

where the function Υ_σ is defined by (cf. Tooper, 1965, Eq. (18))

$$m(r) = 4\,\pi\,\alpha^3\,\rho_c\,\Upsilon_\sigma(\xi); \tag{9}$$

$m(r)$ is the total mass interior to a sphere of radius r (cf. Tooper, 1965, Eq. (12)). In the Newtonian limit $\sigma = 0$, Eqs. (7) and (8) reduce to the classical Lane–Emden equation (Eq. (5) with $i = j = 0$). In the relativistic case $\sigma > 0, \Theta_\sigma$ is the total distortion owing to relativistic effects and can be written as (Geroyannis and Karageorgopoulos, 2014, Eq. (57))

$$\Theta_\sigma = \theta_{00} + \sum_{i=1}^{\infty}\sigma^i\,\theta_{3(i-1)}. \tag{10}$$

The PNA's scheme in Geroyannis and Karageorgopoulos (2014) includes terms of first order in σ; in this case, the sum in Eq. (10) contains the single term $\sigma\,\theta_{30}$. When with infinite terms, the sum in Eq. (10) becomes equal to $\Theta_\sigma - \theta_{00}$. The computational basis of HAS consists in using the complete solution in the relativistic distortion and perturbation terms of up to second order in υ with respect to the rotational distortion.

The initial conditions for solving the differential Eqs. (5), (7), and (8) are written as (cf. Geroyannis and Karageorgopoulos, 2014, Eqs. (26))

$$\begin{aligned}&\theta_{00} = 1, && \frac{d\theta_{00}}{d\xi} = 0, && \text{at } \xi = 0, \\ &\theta_{ij} = 0, && \frac{d\theta_{ij}}{d\xi} = 0, \quad i = 1, 2, \quad j = 0, && \text{at } \xi = 0, \\ &\theta_{ij} = \xi^j, && \frac{d\theta_{ij}}{d\xi} = j\,\xi^{j-1}, \quad i = 1, 2, \quad j \geqslant 2, && \xi \in \delta(0),\end{aligned} \tag{11}$$

where the interval $\delta(0)$ lies in the vicinity of zero, and

$$\Theta_\sigma = 1, \qquad \Upsilon_\sigma = 0. \tag{12}$$

2.3. The complex-plane strategy

Eq. (5) yields for $i = j = 0$ the classical Lane–Emden equation, which, integrated along a prescribed interval $\mathbb{I}_\xi = [\xi_{\text{start}} = 0,\ \xi_{\text{end}}] \subset \mathbb{R}$ with initial conditions (11a, b) gives the Lane–Emden function $\theta_{00}[\mathbb{I}_\xi \subset \mathbb{R}] \subset \mathbb{R}$. To avoid the indeterminate form θ'_{00}/ξ at the origin, we start integration at a point $\xi_{\text{start}} = \xi_0$ close to the origin. Since ξ_0 is small, the initial conditions (11a, b) are valid at the starting point ξ_0 as well. So, the integration interval becomes $\mathbb{I}_{\xi 0} = [\xi_0,\ \xi_{\text{end}}] \subset \mathbb{R}$.