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Decoupling solution moduli of bigravity

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ABSTRACT

A complete classification of exact solutions of ghost-free, massive bigravity is derived which enables the dynamical decoupling of the background, and the foreground metrics. The general decoupling solution space of the two metrics is constructed. Within this branch of the solution space the foreground metric theory becomes general relativity (GR) with an additional effective cosmological constant, and the background metric dynamics is governed by plain GR.

1. Introduction

Recently, a ghost-free [1,2] nonlinear massive gravity theory was constructed in [3,4]. This theory is a nonlinear generalization of [5]. Independently later on, this ghost-free massive gravity with a flat reference metric was also extended to include a general background metric in [6–8]. A ghost-free two-dynamical-metric theory, namely the bigravity as a covering theory of the massive gravity has also been proposed by introducing the dynamics for the background metric [9–12].

In this paper, by referring to the simple observation already pointed out in [10] which leads to a dynamical decoupling of the background [f], and the foreground [g] metrics we will derive the general solutions $f(x^\mu) = F(g(x^\mu), x^\mu)$ which enable the two metrics to be solutions of two disjoint general relativity (GR) theories. This is possible if a portion of the effective energy–momentum tensor entering into the g-metric equations as a course of the interaction Lagrangian between the two metrics vanishes. As the same term also appears in the f-metric equations by being the only contribution, when it vanishes the sets of field equations of the two metrics completely decouple from each other yielding only an algebraic matrix equation which generates this picture. This matrix equation written for f, and g plays the role of a solution ansatz that leads us to a branch of the solution space generated by a Cartesian product of two GR's. This matrix ansatz equation will be at the center of our analysis. In the following, we will derive the general solutions of this cubic matrix equation when none of the parameters of the theory vanish. Thus, we will be able to give a complete description of the solution space $\Gamma[f, g]$ whose elements lead to this dynamical decoupling of the two metric sectors. We will also show that, the classification scheme of the analytically available solutions $\{f, g\}$ admits a similarity equivalence class structure.

In Section one, following a summary of the bigravity dynamics we will obtain the decoupling ansatz matrix equation we have mentioned above. Then, in the next section, we will derive the general solutions of this cubic matrix equation for generic constant coefficients. Since the coefficients in the actual equation are functions of the elementary symmetric polynomials of the solutions themselves a more refining analysis is needed. Therefore, later on, we will present a parametric derivation which enables us to construct not only the solutions of this involved matrix equation, but also their elementary symmetric polynomials as functions of the parameters of the theory. Subsequently, we will show that a subset of the generic solutions for constant coefficients must be omitted, when one insists on having an entire set of nonzero theory parameters. Besides, some of the generic solutions are forced to yield the same form when they are plugged into the actual matrix equation we have. In Section four, we will also present a formal definition of the decoupling solution space $\Gamma[f, g]$ of the bigravity theory. We will show that, this space contains a major subset that is composed of analytically well-defined similarity equivalence classes of solutions. Finally, in Section five we will explicitly construct the proportional background solutions, and give an example of Friedmann–Lemaitre–Robertson–Walker (FLRW) on FLRW case.

2. The dynamical decoupling

The action for the ghost-free bimetric gravity [9–12] for the foreground, g, and the background, f metrics in the presence of two

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types of matter can be given as

$$S = -\frac{1}{16\pi G}\int dx^4\sqrt{-g}\left[R^g + \Lambda^g - 2m^2\mathcal{L}_{int}(\sqrt{\Sigma})\right] + S_M^g - \frac{\kappa}{16\pi G}\int dx^4\sqrt{-f}\left[R^f + \Lambda^f\right] + \epsilon S_M^f, \quad (2.1)$$

where R^g, R^f, Λ^g, Λ^f are the corresponding Ricci scalars, and the cosmological constants for the two metrics, respectively. S_M^g, S_M^f are the two different types of matter which independently couple to g, and f, respectively. The interaction Lagrangian of the two metrics is

$$\mathcal{L}_{int}(\sqrt{\Sigma}) = \beta_1 e_1(\sqrt{\Sigma}) + \beta_2 e_2(\sqrt{\Sigma}) + \beta_3 e_3(\sqrt{\Sigma}), \quad (2.2)$$

where $\{e_n\}$ are the elementary symmetric polynomials

$$e_1 \equiv e_1(\sqrt{\Sigma}) = tr\sqrt{\Sigma},$$
$$e_2 \equiv e_2(\sqrt{\Sigma}) = \frac{1}{2}\left((tr\sqrt{\Sigma})^2 - tr(\sqrt{\Sigma})^2\right),$$
$$e_3 \equiv e_3(\sqrt{\Sigma}) = \frac{1}{6}\left((tr\sqrt{\Sigma})^3 - 3\,tr\sqrt{\Sigma}\,tr(\sqrt{\Sigma})^2 + 2\,tr(\sqrt{\Sigma})^3\right), \quad (2.3)$$

of the square-root-matrix

$$\sqrt{\Sigma} = \sqrt{g^{-1}f}. \quad (2.4)$$

Likewise in [12] the original interaction terms $\beta_0 e_0 = \beta_0$, and $\beta_4 e_4 = \beta_4 \det\sqrt{\Sigma}$ are trivially plugged into the cosmological constants Λ_g, and Λ_f, respectively. If we demand that Eq. (2.2) gives the Fierz–Pauli form in the weak-field limit then we must have [9]

$$\beta_1 + 2\beta_2 + \beta_3 = -1. \quad (2.5)$$

Varying Eq. (2.1) with respect to g gives the g-equation

$$R^g_{\mu\nu} - \frac{1}{2}R^g g_{\mu\nu} - \frac{1}{2}\Lambda^g g_{\mu\nu} - m^2\mathcal{T}^g_{\mu\nu} = 8\pi G T^g_{M\,\mu\nu}. \quad (2.6)$$

Also, variation with respect to f results in the f-equation

$$\kappa\left[R^f_{\mu\nu} - \frac{1}{2}R^f f_{\mu\nu} - \frac{1}{2}\Lambda^f f_{\mu\nu}\right] - m^2\mathcal{T}^f_{\mu\nu} = \epsilon 8\pi G T^f_{M\,\mu\nu}. \quad (2.7)$$

In these field equations the contributions coming from the interaction term that is given in Eq. (2.2) are the effective energy–momentum tensors

$$\mathcal{T}^g_{\mu\nu} = g_{\mu\rho}\tau^\rho{}_\nu - \mathcal{L}_{int} g_{\mu\nu}, \quad (2.8)$$

and

$$\mathcal{T}^f_{\mu\nu} = -\frac{\sqrt{-g}}{\sqrt{-f}} f_{\mu\rho}\tau^\rho{}_\nu, \quad (2.9)$$

respectively. Here $\{\tau^\rho{}_\nu\}$ are the elements of the matrix τ [12]

$$\tau = \beta_3(\sqrt{\Sigma})^3 - (\beta_2 + \beta_3 e_1)(\sqrt{\Sigma})^2 + (\beta_1 + \beta_2 e_1 + \beta_3 e_2)\sqrt{\Sigma}, \quad (2.10)$$

namely $\tau^\rho{}_\nu \equiv [\tau]^\rho{}_\nu$. Both of the effective energy–momentum tensors must be covariantly constant

$$\nabla^g_\mu(\mathcal{T}^g)^\mu{}_\nu = 0, \quad \nabla^f_\mu(\mathcal{T}^f)^\mu{}_\nu = 0. \quad (2.11)$$

If one of these constraints is satisfied then the other one is automatically satisfied [13,14]. As discussed in [10] if one chooses

$$\tau = 0, \quad (2.12)$$

then dynamically the Eqs. (2.6), and (2.7) decouple from each other. In this case the first of the constraints (2.11) gives

$$\partial_\mu\mathcal{L}_{int} = 0, \quad (2.13)$$

and thus, as the solution we will take

$$\mathcal{L}_{int} = -\frac{1}{2}\tilde{\Lambda}. \quad (2.14)$$

Therefore, from Eqs. (2.8), (2.9) we have

$$\mathcal{T}^g_{\mu\nu} = \frac{1}{2}\tilde{\Lambda} g_{\mu\nu}, \quad \mathcal{T}^f_{\mu\nu} = 0. \quad (2.15)$$

Consequently, the g-equation Eq. (2.6) becomes the usual Einstein equations for g with an additional effective cosmological constant $\tilde{\Lambda}$, whereas the dynamically-disjoint f-equation Eq. (2.7) reduces to be coupling-constant-modified Einstein equations for f. The rest of our analysis will be devoted to find the general solutions of the matrix equation[1]

$$\sqrt{\Sigma}\left(A(\sqrt{\Sigma})^2 + B(\sqrt{\Sigma}) + C\mathbf{1}_4\right) = 0, \quad (2.16)$$

where

$$A = -\beta_3,$$
$$B = \beta_2 + \beta_3 e_1,$$
$$C = -\beta_1 - \beta_2 e_1 - \beta_3 e_2, \quad (2.17)$$

which constitute the effective solution space $\Gamma[g,f]$ of the ghost-free bigravity action (2.1) that enables the above-mentioned dynamical decoupling for the foreground, and the background metrics.

3. The structure of the solution space Γ

Now, let us consider the matrix equation

$$AX^3 + BX^2 + CX = X(AX^2 + BX + C\mathbf{1}_4) = X(X - \lambda_1\mathbf{1}_4)(X - \lambda_2\mathbf{1}_4) = 0, \quad (3.1)$$

for a 4×4 matrix function $X(x^\mu)$. For the following analysis we will disregard the solutions which require either of the β-coefficients to be zero. The characteristic polynomial of any 4×4 matrix X would be the degree-four polynomial

$$P_X(t) = det(t\mathbf{1}_4 - X), \quad (3.2)$$

whose four roots are the eigenvalues of X. We should observe first that if X is a solution of Eq. (3.1) then for any invertible matrix function $P(x^\mu)$, $P^{-1}XP$ is also a solution. Therefore, in order to find the general solutions of Eq. (3.1) it would be sufficient to classify the Jordan canonical forms satisfying (3.1). Our main objective next, will be the classification of the similarity equivalence classes of solutions with respect to their minimum polynomials $m(X)$. The roots of $m(X)$ are the same with the eigenvalues of the various equivalence classes of matrices having that minimum polynomial with differing multiplicities of course. Since Eq. (3.1) is a degree-three polynomial equation when it is factorized, its various factors with degrees smaller than or equal to three will define the minimum polynomials of its 4×4 matrix function solutions. In other words, the solutions can be classified with respect to these minimum polynomial factors.

3.1. The algebraic structure

In the following classification, we will identify the entire set of similarity equivalence classes of solutions satisfying Eq. (3.1) by simply taking the coefficients in Eq. (3.1) to be constants. We will group the solutions with respect to their minimum polynomials and it will be sufficient to determine the Jordan canonical form spectrum of each minimum polynomial which corresponds to some combination of the factors in Eq. (3.1).

[1] We will prefer to work with the negative of the Eq. (2.12) in accordance with the massive gravity formalism [15,16], and for future relevance.