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The focusing effect of *P*-wave in the Moon's and Earth's low-velocity core. Analytical solution

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ABSTRACT

The important aspect in the study of the structure of the interiors of planets is the question of the presence and state of core inside them. While for the Earth this task was solved long ago, the question of whether the core of the Moon is in a liquid or solid state up to the present is debatable up to present. If the core of the Moon is liquid, then the velocity of longitudinal waves in it should be lower than in the surrounding mantle. If the core is solid, then most likely, the velocity of longitudinal waves in it is higher than in the mantle. Numerical calculations of the wave field allow us to identify the criteria for drawing conclusions about the state of the lunar core.

In this paper we consider the problem of constructing an analytical solution for wave fields in a layered sphere of arbitrary radius. A stable analytic solution is obtained for the wave fields of longitudinal waves in a three-layer sphere. Calculations of the total wave fields and rays for simplified models of the Earth and the Moon with real parameters are presented. The analytical solution and the ray pattern showed that the low-velocity cores of the Earth and the Moon possess the properties of a collecting lens. This leads to the emergence of a wave field focusing area. As a result, focused waves of considerable amplitude appear on the surface of the Earth and the Moon. In the Earth case, they appear before the first *PKP*-wave arrival. These are so-called "precursors", which continue in the subsequent arrivals of waves. At the same time, for the simplified model of the Earth, the maximum amplitude growth is observed in the 147-degree region. For the Moon model, the maximum amplitude growth is around 180°.

1. Introduction

The important aspect in the study of the structure of the interiors s of planets is the question of the presence and state of cores inside them. While for the Earth this task was solved long ago, the question of whether the core of the Moon is in a liquid or solid state up to the present is debatable up to present. If the core of the Moon is liquid, then the velocity of longitudinal waves in it should be lower than in the surrounding mantle. If the core is solid, then most likely, the velocity of longitudinal waves in it is higher than in the mantle. Numerical calculations of the wave field allow us to identify the criteria for drawing conclusions about the state of the lunar core.

In this paper we consider the problem of constructing a stable analytic solution for wave fields in a layered sphere of arbitrary size. After the Fourier- Legendre transformations, the statement of the problem reduces to the consideration of a two-parameter family of boundary-value problems for ordinary differential equations. The solution of the latter problem in each spherical layer is in the form of a linear combination of Bessel functions (Tikhonov and Samarskii, 1997). The unknown coefficients are determined from known conjugation conditions on the boundary of spherical layers. As a result, a matrix system of linear equations is obtained for their determination. For a small number of layers, its solution can be obtained in explicit form. Since Bessel functions of different types tend to zero and infinity rapidly, uncertainty arises in the solution. And the more the radius of the sphere in relative values (wavelengths), the faster it arises. In this situation, computer calculations become unstable. To construct a stable solution, it is proposed to use the classic asymptotic of Bessel functions (Korneev and Johnson, 1993). In the article (Fatyanov, 2016) it is shown that the classical asymptotic behavior of Bessel functions gives an error in the solution. To construct the solution, we use the new asymptotes of cylindrical functions obtained in the article (Fatyanov, 2016). This gives a stable analytical solution for

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wave fields in an inhomogeneous sphere of arbitrary size.

The results of calculations of wave fields for simplified models of the Moon and the Earth are presented. The Moon model consists of a mantle and a low-velocity core, which implies its liquid state. The Earth model consists of a mantle, an external liquid core and an internal solid core with real parameters (Burmin, 2004). As a result, in the interiors of the Moon and the Earth the area of focusing of the wave field appears. This is due to the fact that the low-speed core of the Earth and possibly the core of the Moon have the properties of a collecting lens. For the Earth, the focused waves go to the surface before the first arrival of the *PKP* wave. In optics the fact that spherical bodies possess the properties of a collecting lens is well known (Kravzov and Orlov, 1980). It turns out that in seismology this phenomenon also exists.

2. Formulation of the problem

The mathematical statement of the problem of modeling the *P*-wave is formulated in a spherical coordinate system $(0 \le r \le R_1,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi)$as follows: define a function $u(r,\theta,\phi,t)$ from equation

$$\frac{1}{v^2(r)}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + f_r \times f(t) \tag{1}$$

with initial conditions and boundary data

$$u = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial r}\right|_{r=R_1} = 0. \tag{2}$$

In (1), (2) R_1 – the radius of the sphere, f_r – is the source function over space, $f(t)$ is the source function with respect to time t.

At boundaries $r = R_j$ where the velocity of longitudinal waves *v(r)* suffers a discontinuity, known conjugation conditions are introduced (Tikhonov and Samarskii, 1997):

$$[u] = \left.\left[\frac{\partial u}{\partial r}\right]\right|_{r=R} = 0. \tag{3}$$

3. Analytical solution

In the case of a concentrated action $f_r = \delta(r-d)\frac{\delta(\theta)}{d^2\sin\theta}$ applied at a point $r = d$and $\theta = 0$ displacement field $u(r,\theta,t)$ independent of the coordinate ϕ is excited. At the first stage the solution is searched in the form of Fourier decomposition - Legendre in the variables (θ,t)

$$u(r,\theta,t) = \frac{1}{2T}\frac{1}{\sqrt{r}}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}u(r,k,\omega_n)\exp(i\omega_n t)P_k(\cos\theta). \tag{4}$$

Here $P_k(x)(-1 \le x \le 1)$, is the Legendre polynomial, $\omega_n = n\pi/T$.

The coefficients of expansion are determined by the formula:

$$u(r,k,\omega_n) = \frac{2k+1}{2}\int_{-1}^{1}\int_0^\infty P_k(\cos\theta)u(r,\theta,t)\exp(-i\omega_n t)d\cos\theta dt. \tag{5}$$

As a result, problem (1)–(2) is reduced to a two-parameter family (k, ω_n) of boundary-value problems in each spherical layer $R_{j+1} < r < R_j$ (Fatyanov, 1981). To reduce the recording, nonessential variables are denoted by the letter c, and inessential indexes are omitted

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{(k+0.5)^2}{r^2}u + c\times\delta(r-d)F(\omega) = -\frac{\omega^2}{c^2}u, \quad \left.\frac{du}{dr} - \frac{0.5}{r}u\right|_{r=R_1} = 0. \tag{6}$$

The conjugation conditions (3) take the form:

$$[u] = \left.\left[\frac{\partial u}{\partial r} - \frac{0.5}{r}u\right]\right|_{r=R} = 0. \tag{7}$$

The paper deals with the case of a three-layered sphere. The boundaries are located at distances $r = R_2, R_3$ from the center of the sphere. The velocities of longitudinal waves in spherical layers are v_1, v_2, v_3. The solution in each spherical layer is defined as a linear combination of Bessel functions (Tikhonov and Samarskii, 1997).

$$u(r,k,\omega) = c_1 J_{k+0.5}(\omega r/v_1) + c_2 J_{-k-0.5}(\omega r/v_1), \quad R_2 < r < R_1 \tag{8}$$

$$u(r,k,\omega) = c_3 J_{k+0.5}(\omega r/v_2) + c_4 J_{-k-0.5}(\omega r/v_3), \quad R_3 < r < R_2$$

Moreover, in the inner spherical shell (containing a center of the sphere)

$$u(r,k,\omega) = c_5 J_{k+0.5}(\omega r/v_3) \tag{9}$$

The unknown coefficients c_1,c_2,c_3,c_4,c_5 are determined from the conjugation conditions on the boundaries of the velocity discontinuity v (7). In the case when the source is located on the surface of the sphere ($r = R_1$) we obtain a system of 5 equations with five unknowns:

$$c_1 J_v\left(\frac{\omega}{v_1}R_2\right) + c_2 J_{-v}\left(\frac{\omega}{v_1}R_2\right) - c_3 J_v\left(\frac{\omega}{v_2}R_2\right) - c_4 J_{-v}\left(\frac{\omega}{v_2}R_2\right) = 0,$$

$$c_1\left\langle\frac{\omega}{v_1}J_v^{'}\left(\frac{\omega}{v_1}R_2\right) - \frac{0.5}{R_2}J_V\left(\frac{\omega}{v_1}R_2\right)\right\rangle + c_2\left\langle\frac{\omega}{v_1}J_{-v}^{'}\left(\frac{\omega}{v_1}R_2\right) - \frac{0.5}{R_2}J_{-V}\left(\frac{\omega}{v_1}R_2\right)\right\rangle + $$
$$+ c_3\left\langle\frac{\omega}{v_2}J_v^{'}\left(\frac{\omega}{v_2}R_2\right) - \frac{0.5}{R_2}J_V\left(\frac{\omega}{v_2}R_2\right)\right\rangle + c_4\left\langle\frac{\omega}{v_2}J_{-v}^{'}\left(\frac{\omega}{v_2}R_2\right) - \frac{0.5}{R_2}J_{-V}\left(\frac{\omega}{v_2}R_2\right)\right\rangle = 0$$

$$c_3 J_v\left(\frac{\omega}{v_2}R_3\right) + c_4 J_{-v}\left(\frac{\omega}{v_2}R_3\right) - c_3 J_v\left(\frac{\omega}{v_3}R_3\right) = 0, \tag{10}$$

$$c_3\left\langle\frac{\omega}{v_2}J_v^{'}\left(\frac{\omega}{v_2}R_3\right) - \frac{0.5}{R_3}J_V\left(\frac{\omega}{v_2}R_3\right)\right\rangle + c_4\left\langle\frac{\omega}{v_2}J_{-v}^{'}\left(\frac{\omega}{v_2}R_3\right) - \frac{0.5}{R_3}J_{-V}\left(\frac{\omega}{v_2}R_3\right)\right\rangle + $$
$$+ c_5\left\langle\frac{\omega}{v_3}J_v^{'}\left(\frac{\omega}{v_3}R_3\right) - \frac{0.5}{R_2}J_V\left(\frac{\omega}{v_3}R_3\right)\right\rangle = 0,$$

$$c_1\left\langle\frac{\omega}{v_1}J_v^{'}\left(\frac{\omega}{v_1}R_1\right) - \frac{0.5}{R_1}J_V\left(\frac{\omega}{v_1}R_1\right)\right\rangle + c_2\left\langle\frac{\omega}{v_1}J_{-v}^{'}\left(\frac{\omega}{v_1}R_1\right) - \frac{0.5}{R_1}J_{-V}\left(\frac{\omega}{v_1}R_1\right)\right\rangle = c\times F(\omega).$$

In system (10) $v = k+0.5$.

From (10) we find a solution in the spectral region for a three-layered sphere on its surface.

$$u(r,k,\omega)|_{r=R_1} = \frac{J_v\left(\frac{\omega}{v_1}R_1\right) + d_5 J_{-v}\left(\frac{\omega}{v_1}R_1\right)}{p_1 + d_5 p_2}cF(\omega). \tag{11}$$

In (11) the following notation is used:

$$p_1 = \frac{\omega}{v_1}J_v^{'}\left(\frac{\omega}{v_1}R_1\right) - \frac{0.5}{R_1}J_V\left(\frac{\omega}{v_1}R_1\right), \quad p_2 = \frac{\omega}{v_1}J_{-v}^{'}\left(\frac{\omega}{v_1}R_1\right) - \frac{0.5}{R_1}J_{-V}\left(\frac{\omega}{v_1}R_1\right)$$

$$d_5 = -\frac{d_1 d_2 - d_4 J_v\left(\frac{\omega}{v_1}R_2\right)}{d_1 d_3 - d_4 J_{-v}\left(\frac{\omega}{v_1}R_2\right)},$$

$$d_1 = J_v\left(\frac{\omega}{v_1}R_2\right) - \frac{b_1}{b_2}J_{-v}\left(\frac{\omega}{v_1}R_2\right), \quad d_2 = \frac{\omega}{v_1}J_v^{'}\left(\frac{\omega}{v_1}R_2\right) - \frac{0.5}{R_2}J_v\left(\frac{\omega}{v_1}R_2\right),$$

$$d_3 = \frac{\omega}{v_1}J_{-v}^{'}\left(\frac{\omega}{v_1}R_2\right) - \frac{0.5}{R_2}J_{-v}\left(\frac{\omega}{v_1}R_2\right),$$