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Long-term resonances between two Jovian exoplanets

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ABSTRACT

Within the plane planetary problem we present two new approaches for the determination of purely resonant eccentricity and semimajor axis variations in terms of simple, closed algebraic relationships. We consider the motion of two Jovian exoplanets in 2:1, 3:1, and 7:4 resonance. Even with initial eccentricities of 0.05, we have found two numerical examples of purely resonant motion of two Jovian exoplanets in 2:1 and 3:1 resonance, fitting throughout the theoretical relationships for over 10^5 revolutions of the outer exoplanet. The maximum eccentricities of the two Jovian exoplanets are <0.15, if the initial ratio of semimajor axes is <0.6992 and the initial eccentricities are ≤0.05. During intervals of negligible secular perturbations, the agreement between theoretical and numerical maximum resonant eccentricity variations is generally much better than within a factor of 2. The theoretical and calculated maximum eccentricity of a Plutino in 2:3 resonance with Neptune is >0.053.

1. Introduction

Mean motion resonances in celestial mechanics have been mainly studied with the Hamiltonian approach (e.g. Brown and Shook, 1933, Secs. 8.15–8.27; Franklin et al., 1984; Murray and Dermott, 1999, Secs. 8.8–8.11; Morbidelli, 2002). Instead, in this paper we directly solve Lagrange's planetary equations for the *plane* planetary problem with two finite masses m_i, $(i = 1, 2)$, orbiting the large mass M, $(m_i \ll M)$ in *prograde* orbits of moderate eccentricity e_i, having semimajor axes a_i, $(\alpha = a_1/a_2)$. This allows a straightforward determination of the resonant variations of eccentricity and semimajor axis of two resonant planets under the form of simple, closed algebraic relationships.

The direct integration of the Lagrangian equations (2.4)–(2.9) for purely resonant motion – as effected in this paper – has been termed the pendulum model by Brown and Shook (1933, Secs. 8.5–8.14) and Murray and Dermott (1999, Secs. 8.4–8.7). This approach to the resonance problem is presented in the textbooks consecutively with the Hamiltonian approach (e.g. Brown and Shook, 1933, Secs. 8.15–8.27; Murray and Dermott, 1999, Secs. 8.8–8.11). A direct comparison of our simple algebraic relationships with Hamiltonian theory seems possible only for the restricted circular three-body problem (see end of Section 5).

Because the resonant variation of semimajor axis is connected to resonant eccentricity variation by elementary relationships (Eqs. (3.9), (3.24), and (4.7)), we concentrate on the variation of eccentricity as a function of the resonance angle

$$\varphi_{k_1k_2} = j_2\lambda_2 - j_1\lambda_1 - k_1\varpi_1 - k_2\varpi_2. \tag{1.1}$$

The integers j_i, k_i, $(i = 1, 2)$ are connected by d'Alembert's rule $j_2 - j_1 - k_1 - k_2 = 0$ (e.g. Murray and Dermott, 1999, Eq. (6.140)), where $j_2 > j_1 \geq 1$ and $k_1 + k_2 \geq 1$, $(k_i \geq 0)$. The mean longitudes of the two planets are denoted by λ_i and their pericenters by ϖ_i.

2. Basic equations

The disturbing functions R_i of the two planets are approximated only by the sum of secular and resonant perturbations R_s and R_r:

$$R_1 = (Gm_2\alpha/a_1)(R_s + R_r), \quad R_2 = (Gm_1/a_2)(R_s + R_r). \tag{2.1}$$

The gravitational constant is denoted by G. The secular and resonant parts are to the lowest order in the eccentricities equal to

$$R_s = b_0^{(1/2)}/2 + K(e_1^2 + e_2^2) + Le_1e_2\cos(\varpi_2 - \varpi_1), \tag{2.2}$$

$$R_r = \sum_{k_1=0}^{j_2-j_1} B_{k_1k_2}e_1^{k_1}e_2^{k_2}\cos\varphi_{k_1k_2} \quad (k_2 = j_2 - j_1 - k_1). \tag{2.3}$$

$K(\alpha)$ and $L(\alpha)$ are the two principal secular coefficients, $B_{k_1k_2} = B_{k_1k_2}(\alpha)$ are the resonance coefficients, and $b_k^{(1/2)}$ denotes the

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Laplace coefficient of index k and order 1/2. The relevant Lagrange equations are (e.g. Brouwer and Clemence, 1961, pp. 284–286; Stumpff, 1965, Sec. 133; Murray and Dermott, 1999, Secs. 6.8, 8.4):

$$\dot{n}_1 = -(3n_1/2a_1)\dot{a}_1 = -\left(3/a_1^2\right)\partial R_1/\partial\lambda_1 = -3(m_2\alpha/M)j_1 n_1^2 \sum_{k_1=0}^{j_2-j_1} B_{k_1k_2} e_1^{k_1} e_2^{k_2} \sin\varphi_{k_1k_2} = \dot{n}_1^{(r)}, \tag{2.4}$$

$$\dot{n}_2 = -(3n_2/2a_2)\dot{a}_2 = -\left(3/a_2^2\right)\partial R_2/\partial\lambda_2 = 3(m_1/M)j_2 n_2^2 \sum_{k_2=0}^{j_2-j_1} B_{k_1k_2} e_1^{k_1} e_2^{k_2} \sin\varphi_{k_1k_2} = \dot{n}_2^{(r)}, \tag{2.5}$$

$$\dot{e}_1 = -\left(1/a_1^2 n_1 e_1\right)\partial R_1/\partial\varpi_1 = (m_2\alpha n_1/M)\left[-Le_2\sin(\varpi_2-\varpi_1) - \sum_{k_1=1}^{j_2-j_1} k_1 B_{k_1k_2} e_1^{k_1-1} e_2^{k_2}\sin\varphi_{k_1k_2}\right] = \dot{e}_1^{(s)} + \dot{e}_1^{(r)}, \tag{2.6}$$

$$\dot{e}_2 = -\left(1/a_2^2 n_2 e_2\right)\partial R_2/\partial\varpi_2 = (m_1 n_2/M)\left[Le_1\sin(\varpi_2-\varpi_1) - \sum_{k_2=1}^{j_2-j_1} k_2 B_{k_1k_2} e_1^{k_1} e_2^{k_2-1}\sin\varphi_{k_1k_2}\right] = \dot{e}_2^{(s)} + \dot{e}_2^{(r)}, \tag{2.7}$$

$$\dot{\varpi}_1 = \left(1/a_1^2 n_1 e_1\right)\partial R_1/\partial e_1 = (m_2\alpha n_1/M)\left[2K + L(e_2/e_1)\cos(\varpi_2-\varpi_1) + \sum_{k_1=1}^{j_2-j_1} k_1 B_{k_1k_2} e_1^{k_1-2} e_2^{k_2}\cos\varphi_{k_1k_2}\right] = \dot{\varpi}_1^{(s)} + \dot{\varpi}_1^{(r)}, \tag{2.8}$$

$$\dot{\varpi}_2 = \left(1/a_2^2 n_2 e_2\right)\partial R_2/\partial e_2 = (m_1 n_2/M)\left[2K + L(e_1/e_2)\cos(\varpi_2-\varpi_1) + \sum_{k_2=1}^{j_2-j_1} k_2 B_{k_1k_2} e_1^{k_1} e_2^{k_2-2}\cos\varphi_{k_1k_2}\right] = \dot{\varpi}_2^{(s)} + \dot{\varpi}_2^{(r)}. \tag{2.9}$$

The superscripts (s) and (r) indicate secular and resonant variations respectively, and the mean motion is $n_i(t) = [G(M+m_i)/a_i^3(t)]^{1/2} \simeq (GM/a_i^3)^{1/2} \simeq$ const. The classical mean longitude at epoch ϵ_i is connected to the modified mean longitude at epoch ϵ_i^* by

$$\lambda_i = n_i t + \epsilon_i = \int_0^t n_i\,dt + \epsilon_i^*. \tag{2.10}$$

There is $\lambda_i(0) = \epsilon_i(0) = \epsilon_i^*(0)$ and $\dot{\epsilon}_i^* = (e_i/2a_i^2 n_i)\,\partial R_i/\partial e_i = e_i^2\dot{\varpi}_i/2 = O(e_i\dot{e}_i) \ll n_i$, since $m_i/M,\ e_i \ll 1$ and $n_i \simeq 1$ if $G, M, a_i = 1$. Thus, $\dot{\epsilon}_i^* \ll n_i,\ \dot{e}_i,\ \dot{\varpi}_i$, and we can write $\dot{\lambda}_i = n_i + \dot{\epsilon}_i^* \simeq n_i$. For the considered modest eccentricities ($e_i \lesssim 0.2$), the maximum relative resonant variation of semimajor axes is small [$\Delta a_i/a_i \ll 1$, Eq. (3.24)], so we always approximate α, n_i, K, L, $B_{k_1k_2}$ on the right-hand sides of Eqs. (2.4)–(2.9) by constants.

The net effect of purely resonant perturbations can be evidenced by neglecting in Eqs. (2.4)–(2.9) the secular eccentricity perturbations ($\dot{e}_i \approx \dot{e}_i^{(r)}$, $|\dot{e}_i^{(s)}| \ll |\dot{e}_i^{(r)}|$), and by approximating the secular pericenter perturbation $\dot{\varpi}_i^{(s)}$ with the constant K_i from Eq. (2.11). These two approximations subsist in the following two cases:

(i) In the circular restricted three-body problem ($m_1 = 0$, $e_2 \to 0$ or $m_2 = 0$, $e_1 \to 0$), there is rigorously $\dot{e}_i^{(s)} = 0$, $\dot{\varpi}_i^{(s)} =$ const, and merely the $(k_1, k_2) = (j_2 - j_1, 0)$ or $(0, j_2 - j_1)$ resonance survives in the sums from Eqs. (2.4) to (2.9).

(ii) If *secular and resonant* eccentricity changes are comparable (as in our numerical examples from Figs. 3, 6, and 8), the secular period of eccentricity variations turns out to be much longer than the resonance period. Therefore, during a single resonance period, the secular eccentricity variations are much smaller than the resonant ones ($|\dot{e}_i^{(s)}| \ll |\dot{e}_i^{(r)}|$), at least for our examples. For this reason, and because in addition we calculate the theoretical resonant eccentricity changes from Table 2 if $\varpi_2 - \varpi_1 \approx \pi$, $\sin(\varpi_2 - \varpi_1) \approx 0$, we neglect $\dot{e}_i^{(s)}$ with respect to $\dot{e}_i^{(r)}$ in Eqs. (3.3), (3.4), and (4.4).

Concerning the secular pericenter variations $\dot{\varpi}_i^{(s)}$ from Eqs. (2.8) and (2.9), we approximate $\cos(\varpi_2 - \varpi_1)$ during a single resonance period by a constant, as argued previously. And because, according to Eqs. (3.10), (4.6), and (4.8), resonant eccentricity variations occur concomitantly, we may also take $e_1/e_2 \approx$ const. Hence, we barely match the L-term of $\dot{\varpi}_i^{(s)}$ during one resonance period by a constant, adding it to the constant K-term:

$$\dot{\varpi}_1^{(s)} = (m_2\alpha n_1/M)[2K + L(e_2/e_1)\cos(\varpi_2-\varpi_1)] \approx K_1 = \text{const},$$
$$\dot{\varpi}_2^{(s)} = (m_1 n_2/M)[2K + L(e_1/e_2)\cos(\varpi_2-\varpi_1)] \approx K_2 = \text{const}. \tag{2.11}$$

In Section 3 we solve with the previously mentioned approximations the set of Lagrangian equations for a single specific resonance: $k_i =$ const. And in Section 4 we consider concomitantly all possible (k_1, k_2) combinations of a certain $j_2 : j_1$ resonance with the additional approximations $e_i \approx e_{i0} =$ const, $\dot{\varpi}_i \approx K_i$, and $\varpi_2 - \varpi_1 \approx 0$ or π, $(0 \le \varpi_2 - \varpi_1 < 2\pi)$.

3. Resonant motion for a single specific (k_1, k_2) resonance

With the approximations $k_i =$ const, $\dot{e}_i = \dot{e}_i^{(r)}$, and $\dot{\varpi}_i = K_i + \dot{\varpi}_i^{(r)}$, the basic Lagrangian equations (2.4)–(2.9) simplify to

$$\dot{n}_1 = -3(m_2\alpha/M)j_1 n_1^2 B_{k_1k_2} e_1^{k_1} e_2^{k_2}\sin\varphi_{k_1k_2}, \tag{3.1}$$

$$\dot{n}_2 = 3(m_1/M)j_2 n_2^2 B_{k_1k_2} e_1^{k_1} e_2^{k_2}\sin\varphi_{k_1k_2}, \tag{3.2}$$

$$\dot{e}_1 = -(m_2\alpha/M)k_1 n_1 B_{k_1k_2} e_1^{k_1-1} e_2^{k_2}\sin\varphi_{k_1k_2}, \tag{3.3}$$

$$\dot{e}_2 = -(m_1/M)k_2 n_2 B_{k_1k_2} e_1^{k_1} e_2^{k_2-1}\sin\varphi_{k_1k_2}, \tag{3.4}$$

$$\dot{\varpi}_1 = K_1 + (m_2\alpha/M)k_1 n_1 B_{k_1k_2} e_1^{k_1-2} e_2^{k_2}\cos\varphi_{k_1k_2}, \tag{3.5}$$

$$\dot{\varpi}_2 = K_2 + (m_1/M)k_2 n_2 B_{k_1k_2} e_1^{k_1} e_2^{k_2-2}\cos\varphi_{k_1k_2}. \tag{3.6}$$

In addition to the previously mentioned circular restricted problem, the disregard of interactions between different (k_1, k_2) resonances, i.e. the present consideration of only one specific resonance also subsists if different resonance angles $\varphi_{k_1k_2}$ are widely separated or if resonance takes place mainly with a single dominant resonance angle, as usually occurs with asteroids (Murray and Dermott, 1999, p. 374, 390). The present approximation $k_i =$ const may be also appropriate for first order resonances ($j_2 - j_1 = k_1 + k_2 = 1$), when in Eqs. (2.4) and (2.5) only one of the two resonant terms needs to be neglected, while the resonant equations for $\dot{e}_i^{(r)}$, $\dot{\varpi}_i^{(r)}$ are identical among Eqs. (2.6)–(2.9) and (3.3)–(3.6).

We assume throughout that the resonance condition (e.g. Murray and Dermott, 1999, Eq. (6.159))