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On the solutions of field equations due to rotating bodies in General Relativity

Vikas Kumar[a,*], Lakhveer Kaur[b]

[a] Department of Mathematics, D.A.V. College, Pundri, Haryana 136042, India
[b] Department of Mathematics, Jaypee Institute of Information Technology, A-10, Sector-62, Noida, Uttar Pradesh 201309, India

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Abstract

A metric, describing the field due to bodies in stationary rotation about their axes and compatible with a stationary electromagnetic field, has been studied in present paper. Using Lie symmetry reduction approach we have herein examined, under continuous groups of transformations, the invariance of field equations due to rotation in General Relativity, that are expressed in terms of coupled system of partial differential equations. We have exploited the symmetries of these equations to derive some ansätz leading to the reduction of variables, where the analytic solutions are easier to obtain by considering the optimal system of conjugacy inequivalent subgroups. Furthermore, some solutions are considered by using numerical methods due to complexity of reduced ordinary differential equations.

Keywords: General Relativity; Electromagnetic field; Lie symmetry method; Exact solutions.

1. Introduction

General Relativity describes phenomena on all scales in the Universe, from compact objects such as black holes, neutron stars, and supernovae to large-scale structure formations such as those involved in creating the distribution of clusters of galaxies. For many years, physicists, astrophysicists and mathematicians have striven to develop techniques for unlocking the secrets contained in Einstein's theory of gravity. More recently, solutions of Einstein field equations have added their expertise to the endeavor. Those who study these objects face a daunting challenge that the equations are among the most complicated in mathematical physics. Together, they form a set of coupled, nonlinear, hyperbolic-elliptic partial differential equations that contain many thousands of terms.

The gravitational field due to a rotating body was first attempted by Thirring who used Einstein field equations in the linear approximation and showed that a rotating thin spherical shell produces near its centre forces analogous to the Coriolis and centrifugal forces of classical machines. Later on this work has been revised by Pirani [23] who supplemented the energy tensor of incoherent material by a term representing the elastic interaction between the particles of the shell. Bach considered the field due to a slowly rotating sphere by successive approximations taking the Schwarzschild solution as his zeroth approximation. Special cases of stationary fields has been considered

* Corresponding author.
E-mail address: davcpundri@kuk.ac.in (V. Kumar).

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by Lanczos [16] and applied its results to cosmological problems.

Lewis [17] found the field due to a rotating infinite cylinder and thus obtained two different methods of successive approximations for constructing solutions of a more general type which behave in an assigned manner at infinity and on a surface of revolution enclosing the rotating matter to which the field is due. Clark tried to solve the empty gravitational field equations, using succussive approximations, with forms of $g_{\mu\nu}$ appropriate to the gravitational field of s rotating body. This introduction provides a sample of the idea that these equations have been a subject of extensive and intensive study both by mathematicians and physicists. For the detail study of exact solutions of Einstein field equations, the reader may refer to Stephani et al. [24]. Recent years have been devoted to studying the field equations of General Relativity for their solutions [1–3,5–7,9,11,12,14,19,20], these solutions are important in the sense that they represent the physical models in analytic manner.

In the present paper, we have considered a metric [17] which is supposed to describe the field due to bodies in stationary rotation. Further in this case we furnished a consistent set of partial differential equations for determining $g_{\mu\nu}$ in empty space time. It is shown that by using the selective form of $g_{\mu\nu}$, the problem of solving four equations in three unknowns has been reduced to a system of two partial differential equations in two unknowns and then Lie group analysis is applied to generate the various symmetries of this coupled system of partial differential equations, which are then used to identify the associated basic vector fields of the optimal system for systematic study of the group invariant solutions admitted by the system.

2. Nature of field equations

The following metric described the field due to bodies in stationary rotation about their axes:

$$ds^2 = -\exp(2\lambda)(d\rho^2 + dz^2) - Cd\phi^2 + Ddt^2 + 2Ed\phi dt, \tag{2.1}$$

where λ, C, D and E are functions of ρ and z only.

Following Lewis [17], we have made use of canonical coordinates in the sense of Weyl. The choice of these coordinates is possible only in matter-free space as it can be easily be verified by a procedure similar to that of Synge. Consequently in domains occupied by matter the canonical coordinates cannot be used.

In canonical coordinates we have

$$CD + E^2 = \rho^2, \tag{2.2}$$

and therefore the expressions for Einstein tensor are given by

$$\begin{aligned}
G_{11} &= -G_{22} = -\frac{\lambda_1}{\rho} - \frac{C_1D_1 + E_1^2 - C_2D_2 - E_2^2}{4\rho^2},\\
G_{33} &= \frac{\exp(-2\lambda)}{2}\bigg(-2C(\lambda_{11}+\lambda_{22}) + C_{11} + C_{22} - \frac{C_1}{\rho}\\
&\quad + \frac{3C}{2\rho^2}(C_1D_1 + E_1^2 + C_2D_2 + E_2^2)\bigg),\\
G_{44} &= \frac{\exp(-2\lambda)}{2}\bigg(2D(\lambda_{11}+\lambda_{22}) - D_{11} - D_{22} + \frac{D_1}{\rho}\\
&\quad - \frac{3D}{2\rho^2}(C_1D_1 + E_1^2 + C_2D_2 + E_2^2)\bigg),\\
G_{34} &= \frac{\exp(-2\lambda)}{2}\bigg(-2E(\lambda_{11}+\lambda_{22}) - E_{11} - E_{22} + \frac{E_1}{\rho}\\
&\quad - \frac{3E}{2\rho^2}(C_1D_1 + E_1^2 + C_2D_2 + E_2^2)\bigg),\\
G_{12} &= -\frac{\lambda_2}{\rho} - \frac{C_1D_2 + 2E_1E_2 + C_2D_1}{4\rho^2},
\end{aligned} \tag{2.3}$$

where lower suffixes 1 and 2 after the unknown functions imply partial differentiation with respect to ρ and z respectively.

Now we have considered the determination equation

$$|G_{\mu\nu} - sg_{\mu\nu}| = 0. \tag{2.4}$$

We found that two of the eigenvalues of $G_{\mu\nu}$ with respect to $g_{\mu\nu}$ are given by

$$s_i = \pm\exp(-2\lambda)\left(G_{22}^2 + G_{12}^2\right)^{\frac{1}{2}}, \qquad i = 1,\ 2, \tag{2.5}$$

and the other two are given by following equation

$$s^2 + Rs - \frac{1}{\rho^2}\left(G_{33}G_{44} - G_{34}^2\right) = 0, \tag{2.6}$$

where R is curvature scalar. It is clear from Eqs. (2.5) and (2.6) that, in general, two eigenvalues of $G_{\mu\nu}$ are equal and opposite while other two are different. Therefore the metric (2.1) in canonical coordinates cannot represent a perfect fluid distribution. But if we do not consider the canonical coordinates then all the eigenvalues of Einstein tensor are different in general. Thus in this case metric (2.1) can be utilized to describe the space-time in the domains occupied by matter.

In case of an electromagnetic field, we have $R = 0$, therefore (2.6) gives

$$S_j = \pm\frac{1}{\rho}\sqrt{(G_{33}G_{44} - G_{34}^2)}, \quad j = 3, 4. \tag{2.7}$$