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An inverse problem for the equation of membrane's vibration

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Abstract

A mathematical model for membrane's vibration process is used in this paper. The model is based on seeking a solution of the second-order hyperbolic differential equation. A new inverse problem is set and investigated in two versions. In the first version the known data are as follows: the coefficient defining the phase velocity, the starting data of the Cauchy problem, the Cauchy problem solution on two given planes, derivatives of the solution along the vector being normal to these planes. The challenge has been in localizing the support of the right-hand side of the equation for vibrations. The algorithm permitting to find the bounded domain containing the unknown support was designed. In the second version the algorithm refers to the case where the coefficient defining the phase velocity is unknown but an interval of its possible values is known. A series of runs was performed to illustrate the proposed model.

Keywords: Simulation; Equation of membrane's vibration; Integral geometry; Inverse problem.

Introduction, notations and problem statement

This study deals with mathematical modeling of physical processes, in particular, the vibrations of an infinite homogeneous membrane under an external force. The corresponding vibration equation has the form

$$u_{tt} - a^2(u_{x_1x_1} + u_{x_2x_2}) = f(t, x_1, x_2), \ (x_1, x_2) \in R^2, \ t > 0, \quad (1)$$

where $u(t, x_1, x_2)$ is the displacement of the membrane point (x_1, x_2) at the time t relative to the membrane plane; $f(t, x_1, x_2)$, $f \in C^1((0, \infty) \times R^2)$ is the external force applied to the point (x_1, x_2) at the time t, a is the perturbation propagation velocity, with $a \in R, \ a > 0$.

Suppose that the following initial conditions are given:

$$u(0, x_1, x_2) = \varphi(x_1, x_2), \ u_t(0, x_1, x_2) = \psi(x_1, x_2),$$
$$\varphi(x_1, x_2) \in C^2(R^2), \ \psi(x_1, x_2) \in C^1(R^2). \quad (2)$$

Eq. (1) and conditions (2) form a Cauchy problem for the equation of vibrations of an infinite membrane. It is well-known that there exists a unique solution to this problem in the form of Poisson's formula.

It was established, for example, in Ref. [1, p. 705–712] that the following property holds for hyperbolic equations and systems: if the right-hand side is finite

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with respect to x for each fixed t, and the initial conditions are also finite, then the solution of the Cauchy problem turns out to be finite with respect to x for every fixed t. Notice that the same fact easily follows from Poisson's formula for problem (1), (2).

Let us now consider a problem inverse to problem (1), (2). Let us make some additional assumptions for this purpose. The function f is assumed to be independent of t, i.e.,

$$f(t, x_1, x_2) = f_1(x_1, x_2),\ \ f_1 \in C^1(R^2),$$

with f_1 taking only non-negative values, and the set of points in which the function f_1 is positive being a bounded domain G. The functions φ and ψ are assumed to be finite.

In this paper, the notation E refers to a rectangle in the plane $t=0$, with the sides parallel to the coordinate axes Ox_1 and Ox_2. The paper is dedicated to examining the following two problems with the above-made assumptions satisfied.

Problem 1. Having the coefficient a and the functions φ, ψ, $u(t, x_1, 0)$, $u_{x_2}(t, x_1, 0)$, $u(t, 0, x_2)$, $u_{x_1}(t, 0, x_2)$ as the initial data, find a rectangle E containing the domain G, such that each of its sides touches the boundary of the domain G.

The purpose of this problem is to localize the support of the right-hand side of Eq. (1); from a physical standpoint, this means approximately determining where the external force is applied to the membrane.

Let us also consider a more general case when it is only known that the coefficient a belongs to a certain interval $[a_1, a_2]$. The initial data will be assumed to be zero in this case.

Problem 2. Having as the interval $[a_1, a_2]$ for the coefficient a and the functions $u(t, x_1, 0)$, $u_{x_2}(t, x_1, 0)$, $u(t, 0, x_2)$, $u_{x_1}(t, 0, x_2)$ as the initial data and assuming that $\varphi = 0$, $\psi = 0$, find a rectangle E containing the domain G.

Notice that this problem is similar to the previous one, but the investigation is carried out in greater uncertainty. It is implied that upon finding the sought-for rectangle, we should strive to minimize its size.

Theoretical investigation of the problems set

First, let us analyze the problem on localizing the source of membrane perturbation in the case when the wave propagation velocity (coefficient a) is known.

Theorem 1. *If all of the above-mentioned assumptions are satisfied, there exists a unique solution to Problem 1.*

Proof. By substituting

$$w_1 = u_t + a u_{x_1},\ \ w_2 = -a u_{x_2},$$

we obtain a modified problem from problem (1), (2):

$$\frac{\partial w_1}{\partial t} - a\left(\frac{\partial w_1}{\partial x_1} - \frac{\partial w_2}{\partial x_2}\right) = f, \tag{3a}$$

$$\frac{\partial w_2}{\partial t} + a\left(\frac{\partial w_2}{\partial x_1} + \frac{\partial w_1}{\partial x_2}\right) = 0, \tag{3b}$$

$$w_1(0, x_1, x_2) = \psi(x_1, x_2) + a\varphi_{x_1}(x_1, x_2), \tag{4a}$$

$$w_2(0, x_1, x_2) = -a\varphi_{x_2}(x_1, x_2). \tag{4b}$$

We should note that the functions w_1 and w_2 also turn out to be finite with respect to x for each fixed t under the assumptions made.

In order to reduce the number of variables in problem (3), (4) to two, we are going to use the method of plane mean values proposed by Courant [1, p. 705–712]. For this purpose, let us use following property: the integral of the derivative of a finite function with respect to a variable along the numerical axis corresponding to this variable is equal to zero. Following this method, we can integrate the equations of system (3) with respect to the variable x_1 from minus infinity to plus infinity and introduce the following notation:

$$\int_{-\infty}^{\infty} w_1(t, x_1, x_2)dx_1 = v_1(t, x_2),$$

$$\int_{-\infty}^{\infty} w_2(t, x_1, x_2)dx_1 = v_2(t, x_2).$$

Then, using the condition that the functions w_1 and w_2 are finite and, consequently,

$$\int_{-\infty}^{\infty} \frac{\partial w_i(t, x_1, x_2)}{\partial x_1} dx_1 = 0,\ \ i = 1, 2,$$

we obtain the following problem from problem (3), (4):

$$\frac{\partial v_1}{\partial t} + a\frac{\partial v_2}{\partial x_2} = g(t, x_2), \tag{5a}$$

$$\frac{\partial v_2}{\partial t} + a\frac{\partial v_1}{\partial x_2} = 0, \tag{5b}$$

$$v_1(0, x_2) = \int_{-\infty}^{\infty} \psi(x_1, x_2)dx_1, \tag{6a}$$

$$v_2(0, x_2) = \int_{-\infty}^{\infty} -a\varphi_{x_2}(x_1, x_2)dx_1, \tag{6b}$$

where $g(t, x_2) = \int_{-\infty}^{\infty} f(t, x_1, x_2)dx_1$.