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On homogenous solutions of the problem of a rectangular cantilever plate bending

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Abstract

The paper considers the method suggested by Papkovich for rectangular plates and its application for a cantilever plate bending under a uniform load. The required function of the bendings is chosen in the form of a sum of the corresponding beam function and a biharmonic function, which is a series in terms of unorthogonal eigenfunctions of the problem. The eigenfunctions satisfy the homogenous boundary conditions on the longitudinal edges (the clamped and the opposite ones). It is suggested to find series coefficients from the condition of the minimum residuals effect on the corresponding displacements of the transverse edges. It leads to an infinite system of linear algebraic equations for the required coefficients in the complex form. The coefficients of homogenous solutions were found for the cases in which the approximating series contained sequentially 2, 3,...,7 terms. The eigenvalues, the bendings of the edge opposite to the clamped edge, and the bending moments in the clamped section were calculated. Convergence of the reduction method and stability of the computational process were analyzed.

Keywords: Rectangular cantilever plate; Bending; Homogenous solution; Numerical result; Instability of calculations.

Introduction

A rectangular cantilever plate is a computational scheme for many elements of engineering structures. In particular, cutting tools for a number of manufacturing processes are fabricated as rectangular plates rigidly clamped along one edge. Stress–strain analysis of cantilever plates is also used in calculating the strength and stiffness of the individual elements in the constructions of water turbines, ships, planes and missiles. The bending problem of a rectangular cantilever plate has no exact closed solution, and the results obtained by the known approximate solutions have to be analyzed for accuracy. The spread in numerical results from different authors reaches 20%.

The goal of this study is to obtain reliable numerical results for the stress–strain state of these plates by the method of homogenous solutions.

The first study on calculating cantilever plates was carried out by Holl [1], who used the finite difference method (FDM, also called the grid method) for a wide plate with an aspect ratio of 4:1. A concentrated force applied in the center of the free edge opposite to the clamped one served as the load. This method was also used by many authors for different

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types of loads, plate aspect ratios and grid spacings. The finite element method (FEM) was first applied by Zienkiewicz and Cheung [2]; they divided a square plate into nine square elements and considered the case of a uniform load. Other approximate techniques, such as the Rayleigh–Ritz, the Bubnov–Galerkin and the Kantorovich–Vlasov methods, and others, were also used to solve this problem. Study [3] applied the method of infinite superposition of correcting functions in terms of hyperbolic trigonometric series in order to find the deflection function; in the course of the application of the method, all residual errors in the boundary conditions tended to zero, providing an exact solution of the problem in the limit.

Homogenous solutions and relations of generalized orthogonality of elastic rectangular plates

A solution of the biharmonic equation

$$\nabla^2\nabla^2 w = 0 \tag{1}$$

was sought for by Papkovich [4] in the form

$$w(x,y) = \sum_k c_k e^{-\beta_k x} F_k(y), \tag{2}$$

where ∇^2 is the two-dimensional Laplace operator, w is the plate deflection, c_k are the series coefficients, β_k are the eigenvalues, and $F_k(y)$ are the eigenfunctions of the problem. For a plane elasticity problem, w is the Airy function, and in the case of thin plate bending it is the deflection. After sum (2) is substituted into Eq. (1), an ordinary differential equation is obtained for the $F_k(y)$ functions:

$$F_k^{IV} + 2\beta_k^2 F_k'' + \beta_k^4 F_k = 0, \tag{3}$$

whose general solution has the form

$$F_k(y) = A_k \sin\beta_k y + B_k \cos\beta_k y + C_k y cos\beta_k y + D_k y \sin\beta_k y. \tag{4}$$

Let us consider a rectangular plate with the relative dimensions

$$-\gamma/2 \le x \le \gamma/2,\ 0 \le y \le 1$$

where $\gamma = a/b$ is the aspect ratio of the plate.

If we impose the requirement that deflection function (2) of the plate must satisfy the homogenous boundary conditions at the edges $y=0$ and $y=1$, then a transcendental equation whose solution involves the eigenvalues β_k is obtained for each type of these conditions. For example, when the plate is clamped at both edges, the transcendental equation takes the following form:

$$\sin 2\beta_k \pm 2\beta_k = 0$$

(here the plus sign refers to the even functions in expression (4), and the minus sign to the odd ones).

These equations have an infinite number of complex roots forming groups of quartets (some of the roots may be real).

If the eigenvalues are found, then the functions $e^{-\beta_k x}F_k(y)$ are called homogenous solutions.

For a non-orthogonal system of complex functions that are the eigenfunctions of the problem $F_k(y)$, the authors of Refs. [5,6] have established a relation

$$\int_0^1 \left[F''_k(y)F''_s(y) - \beta_k^2\beta_s^2 F_k(y)F_s(y)\right]dy = 0 \tag{5}$$

(at $k \ne s$), which is called the relation of generalized orthogonality.

The coefficients c_k of series (2) must be determined from the boundary conditions at the transverse edges $x = \pm\gamma/2$. Since usually there are two such conditions and one sequence of coefficients c_k, generally, two boundary conditions cannot be strictly satisfied at once. It should be noted here that although the coefficients c_k consist of real and imaginary components, the requirement that the results must be real links them together. However, in the particular case, if the boundary conditions are such that in order to find the coefficients c_k we need to perform, at the edges $x = \pm\gamma/2$, a joint expansion of two different given functions of y into series of the form

$$f_1(y) = \sum_k a_k L_1[F_k(y)], \quad f_2(y) = \sum_k a_k L_2[F_k(y)], \tag{6}$$

(a_k are some unknown complex constants, proportional to c_k) and the functions L_1 and L_2 are expressed by the formulae

$$L_1\left[F_k(y)\right] = F_k(y), \quad L_2\left[F_k(y)\right] = \beta_k^2 F_k(y) \tag{7}$$

or by the formulae

$$L_1\left[F_k(y)\right] = \beta_k F_k(y), \quad L_2\left[F_k(y)\right] = \beta_k^3 F_k(y), \tag{8}$$

then it is possible to simultaneously satisfy two conditions (see, e.g., Ref. [4]); in this case, generalized orthogonal relation (5) is used, and the coefficients a_k are found by applying the Fourier procedure.

Eq. (5) was proved for the case when the edges $y=0$ and $y=1$ are clamped. Prokopov [7] established that Eq. (5) also holds for the free edges, and if one of the edges is clamped and the other is free.