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Exact expression for the magnetic field of a finite cylinder with arbitrary uniform magnetization

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ABSTRACT

An exact analytical expression for the magnetic field of a cylinder of finite length with a uniform, transverse magnetization is derived. Together with known expressions for the magnetic field due to longitudinal magnetization, the calculation of magnetic fields for cylinders with an arbitrary magnetization direction is possible. The expression for transverse magnetization is validated successfully against the well-known limits of an infinitely long cylinder, the field on the axis of the cylinder and in the far field limit. Comparison with a numerical finite-element method displays good agreement, making the advantage of an analytical method over grid-based methods evident.

1. Introduction

Analytic expressions for the magnetic fields produced by inherently magnetic materials or induced in magnetically susceptible materials, are only well-known for some classic textbook cases, such as the field of point multipoles and infinitely long wires carrying a current [1–4]. In the past, many papers on demagnetization factors [5–11] and cylindrical ferromagnets [12–15] have been published. In demagnetization tensors with regard to uniformly magnetized finite cylinders, implicit analytic expressions have been incorporated [16,17]. Kraus [16] applies a magnetic surface charge method using integrals that contain Bessel functions. Tandon et al. [17] and Beleggia et al. [18,19] employ a Fourier transform approach. Herein use is made of a shape function that is equal to the trace of the demagnetization tensor, which connection is difficult to derive from the commonly used magnetic surface charge description. Magnetic fields of complex geometries often can be solved only numerically via finite element methods (F.E.M.) [20,21]. However, the domain discretization inherent to these methods may ultimately lead to numerical inaccuracies, unless expensive higher-order calculations are performed, or the calculation mesh is refined. The analytic modelling of the field has a clear advantage over finite-element methods as the necessary magnetic quantities can be probed at all required coordinates, with minimal computational effort. This is highly useful, for example, when dynamical systems are modelled, such as the movement of magnetic nanoparticles in magnetic field gradient [22,23].

A geometry for which analytical expressions for magnetic quantities are readily available, is an axisymmetric solenoid of finite length [24–27]. Exact expressions for the vector potential $\mathbf{\Phi}$, magnetic flux density $\mathbf{B}$ (with axial and radial components), magnetic force $\mathbf{F} = (\mathbf{m} \cdot \nabla)\mathbf{B}$, where $\mathbf{m}$ is the magnetic dipole moment of the object, and other quantities can be formulated using special functions such as elliptic integrals. The derivation of these expressions usually extends the treatment of a single circular current loop by integrating over a certain length along the symmetry axis of the loop [28,29]. The solenoid field also describes the field of a cylindrical uniform permanent magnet with its magnetization vector $\mathbf{M}$ along the axis of symmetry (longitudinal magnetization). For different magnetization directions, such as $\mathbf{M}$ perpendicular to the axis of symmetry (transverse magnetization), other field equations are required. In the case of transverse magnetization, explicit analytical results are available for an infinite cylinder [2,30], and for the on-axis field of a finite cylinder derived by Wysin [31]. To expand upon these known relations, we have derived an explicit, analytical expression for the magnetic field of a transversely, uniformly magnetized finite cylinder in all spatial field points, inside as well as outside the cylinder. By combining the expression for longitudinal and transverse magnetization we will also demonstrate the possibility of accurately calculating the resulting magnetic field for a cylinder with an arbitrarily chosen magnetization vector.

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The expressions derived here are applied to the modelling of a high-gradient magnetic separation process, using a separation filter comprising many small magnetizable fibres. By combining the local magnetic fields of a large collection of (non-overlapping) cylinders, we aim to calculate the movement of magnetic nanoparticles through such a separation filter [32] and whether, ultimately, the nanoparticles can be trapped by the filter. In this paper, in addition to the calculations for a single cylinder, we explore the possibility to calculate the magnetic field for a combination of multiple cylinders by means of our analytical expressions, which is also relevant for a broad range of other applications [33–35].

2. Preliminary

Consider a circular cylindrical body of radius R and semi-length L, with its centroid at the origin of a cylindrical coordinate system (ρ, φ, z) and its axis aligned with the z-direction (see Fig. 1). A uniform magnetization of the body along an arbitrarily chosen magnetization vector $\mathbf{M}$ can always be decomposed into a longitudinal and transverse component,

$$\mathbf{M} = M_l \hat{\mathbf{z}} + M_t \hat{\rho} \tag{1}$$

In reality, for a magnetizable material, the acquired magnetization will in general not have the same direction as the applied field $\mathbf{H}_{ext}$, as the magnetization vector will rotate to minimize its energy depending on the magnetic susceptibility of the material and the demagnetization factors of the body. The Stoner-Wohlfarth model describes this principle in detail [36,37]. In general, the magnetization is related to the magnetic field $\mathbf{H}$, the magnetic flux density $\mathbf{B}$ and the permeability of vacuum μ_0 through,

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{2}$$

We proceed by restating the known expression for $\mathbf{B}$ for longitudinal magnetization [25,27] and continue by deriving an expression for the case of transverse magnetization. The validity of the equations are tested by determining several limiting cases. By combining Eqs. (1) and (2), the field of a finite cylinder with an off-axis magnetization vector is calculated and these results are compared with numerical calculations. Finally, the applicability of our model to the description of magnetizable cylinders are tested against the results of a finite element method.

3. Longitudinal magnetization (review of past work)

Equations for the field inside and outside a longitudinally magnetized, finite cylinder were first retrieved by Callaghan and Maslen [25]. They obtained their result by considering a finite cylinder as a collection of current loops (i.e. an ideal solenoid). The total magnetization is $M \equiv nI$, with n the number of turns per unit of length and I the current per turn. By applying the Biot–Savart law, the magnetic field can be calculated directly in terms of elliptic integrals. Derby and Olbert [27] revisited the derivation and provided a computationally convenient form using a combination of generalized complete elliptic integrals [38]. They correctly retrieved the field of a current loop in the limit $L \to 0$ and the far-field limit of a point dipole at large distances from the cylinder.

In Derby and Olbert [27] only an integral form of the field equations is given. Here we restate these results in closed form, in terms of elliptic integrals, obtaining equations similar to those for the transverse case presented in the following section.

$$\begin{aligned} B_\rho &= \frac{\mu_0 M R}{\pi}[\alpha_+ P_1(k_+) - \alpha_- P_1(k_-)] \\ B_z &= \frac{\mu_0 M R}{\pi(\rho + R)}[\beta_+ P_2(k_+) - \beta_- P_2(k_-)] \end{aligned} \tag{3}$$

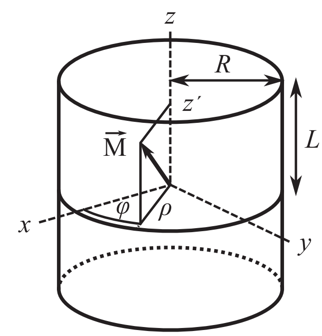

Fig. 1. Schematic representation of a magnetized cylinder of semi-length L and radius R with an arbitrary magnetization vector $\mathbf{M}$. The cylindrical (ρ, φ, z), and Cartesian (x, y, z) coordinate systems are indicated.

where B_ρ and B_z are the radial and axial components of the magnetic flux density, respectively. Two auxiliary functions are defined (see Appendix A) as,

$$\begin{aligned} P_1(k) &= \mathcal{K} - \frac{2}{1-k^2}(\mathcal{K} - \mathcal{E}) \\ P_2(k) &= -\frac{\gamma}{1-\gamma^2}(\mathcal{P} - \mathcal{K}) - \frac{1}{1-\gamma^2}(\gamma^2\mathcal{P} - \mathcal{K}) \end{aligned} \tag{4}$$

and the following shorthand notations will be employed:

$$\begin{aligned} \xi_\pm &= z \pm L \\ \alpha_\pm &= \frac{1}{\sqrt{\xi_\pm^2 + (\rho+R)^2}} \qquad \beta_\pm = \xi_\pm \alpha_\pm \\ \gamma &= \frac{\rho - R}{\rho + R} \qquad k_\pm^2 = \frac{\xi_\pm^2 + (\rho - R)^2}{\xi_\pm^2 + (\rho + R)^2} \end{aligned} \tag{5}$$

The symbols $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{P}$ are used to indicate the evaluation of the complete elliptic integrals of the first, second and third kind, as follows,

$$\begin{aligned} \mathcal{K} &= \mathrm{K}\left(\sqrt{1-k^2}\right) &&= \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-(1-k^2)\sin^2\theta}} \\ \mathcal{E} &= \mathrm{E}\left(\sqrt{1-k^2}\right) &&= \int_0^{\frac{\pi}{2}} d\theta \sqrt{1-(1-k^2)\sin^2\theta} \\ \mathcal{P} &= \Pi\left(1-\gamma^2, \sqrt{1-k^2}\right) &&= \int_0^{\frac{\pi}{2}} \frac{d\theta}{\left(1-(1-\gamma^2)\sin^2\theta\right)\sqrt{1-(1-k^2)\sin^2\theta}} \end{aligned} \tag{6}$$

Note that B_φ is absent in Eq. (3) due to the radial symmetry of the system. A visualization of the magnetic field lines produced by these equations is given in Fig. 2a.

4. Transverse magnetization

To derive the field equations for a transversely magnetized cylinder, we follow the approach of Callaghan and Maslen [25] and Derby and Olbert [27]. We start by choosing a magnetization vector perpendicular to the long axis of the cylinder. A convenient choice is a magnetization along the Cartesian x-axis, $\mathbf{M} = M\hat{\mathbf{x}}$, although any direction in the xy-plane would be suitable for symmetry reasons. Assuming there are no free currents present, the magnetic field can be expressed as the gradient of a magnetostatic scalar potential

$$\mathbf{H} = -\nabla\Phi_m \tag{7}$$

In the following, we derive the exact expression for the potential Φ_m. The components of the $\mathbf{H}$-field can be derived following similar mathematical manipulations, but only the final results will be presented in Section 4.2.