Ain Shams Engineering Journal (2015) 6, 1121–1127

Ain Shams University

Ain Shams Engineering Journal

www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

Numerical treatment of singular perturbation problems exhibiting dual boundary layers

K. Phaneendra *, S. Rakmaiah, M. Chenna Krishna Reddy

Department of Mathematics, University College of Science, Osmania University, Hyderabad, India

Received 1 December 2014; revised 30 January 2015; accepted 22 February 2015
Available online 29 March 2015

KEYWORDS

Singular perturbation problem;
Dual layer;
Fitting factor;
Tridiagonal system;
Discrete invariant imbedding algorithm

Abstract In this paper, we employed a fitted operator finite difference method on a uniform mesh for solving singularly perturbed two-point boundary value problems exhibiting dual boundary layers. In this method, we have extended the Numerov method to the second order singularly perturbed two-point boundary value problem with first order derivative. By using nonsymmetric finite differences for the first order derivative, we have derived the finite difference scheme. A fitting factor is introduced in this finite difference scheme which takes care of the rapid changes that occur in the boundary layer. This fitting factor is obtained from the asymptotic approximate solution of singular perturbations. Discrete invariant imbedding algorithm is used to solve the tridiagonal system of the fitted finite difference method. We have discussed the convergence analysis of the proposed method. Maximum absolute errors of the several numerical examples are presented to illustrate the proposed method.

1. Introduction

Singular perturbation problem now is a maturing mathematical subject with fairly long history and a strong promise for continued important applications throughout science and engineering. A singular perturbation problem is well defined as one in which no single asymptotic expansion is uniformly valid throughout the interval, as the perturbation parameter $\varepsilon \to 0$. Singular perturbation problems arise very frequently in fluid mechanics, fluid dynamics, elasticity, aerodynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography and other domains of the great world of fluid motion. A few notable examples are boundary layer problems, WKB problems, the modeling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers, magneto-hydrodynamics duct problems at high Hartman numbers, etc. Equations of this type typically exhibit solutions with layers; that is, the domain of the differential equation contains narrow regions where the solution derivatives are extremely large. The numerical treatment of singularly perturbed differential equations gives major computational difficulties due to the presence of boundary and/or interior layers. If we apply the existing standard numerical methods for solving these problems, large

* Corresponding author.
E-mail addresses: kollojuphaneendra@yahoo.co.in (K. Phaneendra), rakmajirao@gmail.com (S. Rakmaiah), krishnareddychenna@yahoo.com (M. Chenna Krishna Reddy).

Peer review under responsibility of Ain Shams University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.asej.2015.02.012
2090-4479 © 2015 Faculty of Engineering, Ain Shams University. Production and hosting by Elsevier B.V.

oscillations may arise and pollute the solution in the entire interval because of the boundary layer behavior. Thus more efficient and simpler computational techniques are required to solve singularly perturbed two-point boundary value problems.

The survey papers by Kadalbajoo and Reddy [1], Kadalbajoo and Patidar [2] give an erudite outline of the singular perturbation problems and their treatment on fluid dynamical boundary layers. These survey articles will remain as the most readable sources on singular perturbations. Abrahamsson [3] derived a priori estimates for the solutions of SPPs with a turning point. A set of general sufficient conditions for a uniformly convergent scheme for singularly perturbed turning point problem is obtained by Farrell [4]. Natesan and Ramanujam [5] derived a computational method for the singularly perturbed turning point problem in which exponentially fitted difference schemes are combined with classical numerical methods. Another technique known as initial-value technique was extended in [6] for the singularly perturbed turning point problem in which the numerical solution is obtained by solving suitable initial and terminal value problems. Natesan et al. [7] proposed a parameter uniform numerical method on Shishkin mesh to solve singularly perturbed turning point problems. Miller et al. [8] elucidate the classical schemes on Shishkin meshes to solve singularly perturbed BVPs of convection – diffusion and reaction – diffusion problems subject to Dirichlet boundary conditions.

In this paper, we employed a fitted operator finite difference method on a uniform mesh for solving singularly perturbed two-point boundary value problems exhibiting dual boundary layers. In Section 2, we described the fitted finite difference method by extending the Numerov method to the second order singularly perturbed two-point boundary value problem with first order derivative. In Section 3, we discussed the convergence analysis of the proposed method. To demonstrate the efficiency of the proposed method, numerical experiments are carried out for several test problems and the results are given in Section 4. Finally the discussions and conclusion are given in the last section.

2. Description of the method

Consider singularly perturbed boundary value problems of the form:

$$Ly \equiv \varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), -1 \leqslant x \leqslant 1, \quad (1)$$

with boundary conditions $y(-1) = \alpha$ (2a)

and $y(1) = \beta$ (2b)

where $0 < \varepsilon \ll 1$, α and β are finite constants.

Here, we assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions such that

$$a(0) = 0, \qquad a'(0) \le 0,$$

$$|a(x)| \ge a_0 > 0, \quad \text{for } 0 < x \le 1,$$

$$b(x) \geqslant b_0 > 0, \quad \forall x \in D = [-1, 1],$$

$$|a'(x)| \ge \tfrac{|a'(0)|}{2}, \quad \forall x \in D.$$

With the above assumption, the turning point problem (1)–(2) possesses unique solution exhibiting two boundary layers of exponential type at both end points $x = -1, 1$.

Divide the interval $[-1, 1]$ into N equal parts with mesh size h, i.e., $h = \frac{2}{N}$ and $x_i = -1 + ih$ for $i = 0, 1, \ldots, N$. Let us denote $\frac{N}{2} = l$. Then, divide the interval $[-1, 1]$ into two subintervals $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, l-1$; and $[x_i, x_{i+1}]$ for $i = l+1, l+2, \ldots, N-1$. For dual layer problem, in the interval $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, l-1$ layer exists at left end point and in $[x_i, x_{i+1}]$ for $i = l+1, l+2, \ldots, N-1$ layer is at right end point. Hence, we derive the numerical method for both left-end layer in $[-1, 0]$ and right-end layer in $[0, 1]$ cases.

In the interval $[-1, 0]$, from the theory of singular perturbation is well known that zeroth order asymptotic approximation to the solution of Eq. (1) is (cf. O'Malley [9])

$$y_i \approx y_0(-1 + ih) + (\alpha - y_0(-1)) \exp\left\{-\left(\frac{a(-1)}{\varepsilon}\right)(-1 + ih)\right\}.$$

Therefore

$$\lim_{h\to 0} y_i \approx y_0(-1) + (\alpha - y_0(-1)) \exp\left(-a(-1)\left(\frac{-1}{\varepsilon} + i\rho\right)\right) \quad (3)$$

where $\rho = \frac{h}{\varepsilon}$.

By the Numerov method, we have

$$y_{i-1} - 2y_i + y_{i+1} = \frac{h^2}{12}(y''_{i-1} + 10y''_i + y''_{i+1}) + O(h^6) \quad (4)$$

Now, we extend this method for second order singular perturbation boundary value problem with first order derivative as follows:

From the Eq. (1), we have

$$\varepsilon y''_{i+1} = -a_{i+1}y'^{*}_{i+1} - b_{i+1}y_{i+1} + f_{i+1} \quad (5a)$$

$$\varepsilon y''_i = -a_i y'_i - b_i y_i + f_i \quad (5b)$$

$$\varepsilon y''_{i-1} = -a_{i-1}y'^{*}_{i-1} - b_{i-1}y_{i-1} + f_{i-1} \quad (5c)$$

We approximate y'^{*}_{i+1}, y'^{*}_{i-1} using nonsymmetric finite differences and y'_i by upwind finite difference

$$y'^{*}_{i+1} = \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h} + O(h^2) \quad (6a)$$

$$y'_i = \frac{y_{i+1} - y_i}{h} + O(h) \quad (6b)$$

$$y'^{*}_{i-1} = \frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h} + O(h^2) \quad (6c)$$

Substituting Eqs. (5) and (6) in Eq. (4) and simplifying, we get

$$\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \frac{a_{i-1}}{24h}(-3y_{i-1} + 4y_i - y_{i+1}) + \frac{10a_i}{12h}(y_{i+1} - y_i) + \frac{a_{i+1}}{24h}(y_{i-1} - 4y_i + 3y_{i+1}) + \frac{b_{i-1}}{12}y_{i-1} + \frac{10b_i}{12}y_i + \frac{b_{i+1}}{12}y_{i+1} = \frac{(f_{i-1} + 10f_i + f_{i+1})}{12}$$

Now, introducing the fitting factor $\sigma(\rho)$ (also called artificial viscosity) in the above scheme