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Evolution of weak discontinuity in a van der Waals gas

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Abstract In this article, the Lie symmetries analysis that leaves the system of partial differential equations (PDEs), governed by the one dimensional unsteady flow of an isentropic, inviscid and perfectly conducting compressible fluid obeying the van der Waals equation of state invariant, is presented. Using these symmetries the governing system of PDEs is reduced into system of ordinary differential equations (ODEs). Then the reduced system of ODEs is solved analytically which in turn produces the exact solution for the governing PDEs. Further, the influence of the van der Waals excluded volume in the behavior of evolution of weak discontinuity is studied extensively.

1. Introduction

Many physical phenomena in this universe are modeled by systems of nonlinear hyperbolic PDEs. The explicit determination of exact solutions to such system of nonlinear PDEs of physical interest is an important task. To solve such system of nonlinear PDEs, no general theory is available there and it is also very difficult to systematically construct their exact solutions. Lie group analysis is one of the systematic and most powerful techniques to obtain the exact solutions for such nonlinear system of PDEs (see, [1–4]). This technique has been applied by many researchers to solve different flow phenomena over different geometries. Similarity solutions for three dimensional Euler equations using Lie group analysis is found in [5], whereas the use of same technique to obtain some exact solutions to the ideal magnetogasdynamic equations is described in [6]. Exact solution to axisymmetric flow of shallow water equations by Lie group point transformation is found in [7] whereas in [8], the author derived the self similar solutions for system of PDEs describing a plasma with axial magnetic field (θ-pinch). The work in [9,10] accounts symmetry reductions, group invariant solutions and some exact solutions of (2 + 1)-dimensional Jaulent–Miodek equation. Propagation of weak discontinuities in binary mixture of ideal gases in one-dimensional ideal isentropic magnetogasdynamics has been studied in [11,12] while the interaction of a weak discontinuity wave with the elementary waves for the Euler equations governing the flow of ideal polytropic gases is investigated in [13]. In [14], they found the extensive study on evolution of weak discontinuities in a two-dimensional steady supersonic flow of a non-ideal radiating gas.

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In the present work, Lie group transformations have been used to reduce the governing system of PDEs to a system of ODEs. We solve the reduced system of ODEs and a particular exact solution to the governing system of PDEs is obtained. Further, we discuss the evolution of weak discontinuity in the presence of van der Waals excluded volume.

2. Group analysis

The one dimensional unsteady flow of an isentropic, inviscid and perfectly conducting compressible fluid obeying the equation of state $p = k\left(\frac{\rho}{1-a\rho}\right)^{\gamma}$ can be written as [15]

$$\begin{aligned} &\rho_t + u\rho_x + \rho u_x = 0,\\ &u_t + uu_x + \frac{B}{\rho}B_x + \frac{k\gamma\rho^{\gamma-2}}{(1-a\rho)^{\gamma+1}}\rho_x = 0,\\ &B_t + uB_x + Bu_x = 0, \end{aligned} \tag{1}$$

where ρ, u and B are density, velocity and magnetic field respectively. Here a is the van der Waals excluded volume, k is a positive constant, γ is the adiabatic exponent and the independent variables t and x denote time and space respectively.

Now, we consider Lie group of transformations with the independent variables x and t, and with the dependent variables ρ, u, B for the current problem as

$$\begin{aligned} &t^* = t + \epsilon\phi_1(x,t,\rho,u,B) + O(\epsilon^2), \quad x^* = x + \epsilon\phi_2(x,t,\rho,u,B) + O(\epsilon^2),\\ &\rho^* = \rho + \epsilon\psi_1(x,t,\rho,u,B) + O(\epsilon^2), \quad u^* = u + \epsilon\psi_2(x,t,\rho,u,B) + O(\epsilon^2),\\ &B^* = B + \epsilon\psi_3(x,t,\rho,u,B) + O(\epsilon^2), \end{aligned} \tag{2}$$

where $\phi_1, \phi_2, \psi_1, \psi_2$ and ψ_3 are the generators to be determined such that the system of PDEs (1) remains invariant with respect to the transformations (2) and ϵ is very small group parameter. A straightforward analysis [1], provides us the following infinitesimal transformations:

$$\begin{aligned} &\phi_1 = \alpha_1 + \alpha_2 t, \quad \phi_2 = \alpha_3 + \alpha_4 t + \alpha_2 x, \quad \psi_1 = 0,\\ &\psi_2 = \alpha_4, \quad \psi_3 = 0, \end{aligned} \tag{3}$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are arbitrary constants. The similarity variables can be obtained from the characteristic equations given as below:

$$\frac{dt}{\phi_1} = \frac{dx}{\phi_2} = \frac{d\rho}{\psi_1} = \frac{du}{\psi_2} = \frac{dB}{\psi_3}, \tag{4}$$

i.e.,

$$\frac{dt}{\alpha_1 + \alpha_2 t} = \frac{dx}{\alpha_3 + \alpha_4 t + \alpha_2 x} = \frac{d\rho}{0} = \frac{du}{\alpha_4} = \frac{dB}{0}.$$

Now we consider different cases to obtain the solutions for the governing system of PDEs (1).

Case A: $\alpha_1 = 0$ and $\alpha_2 \neq 0$.

The similarity variable and new dependent variables are

$$\xi = \left(x + \frac{\alpha_3}{\alpha_2}\right) - t\ln t^{\frac{\alpha_4}{\alpha_2}}, \quad \rho = R, \quad u = \frac{\alpha_4}{\alpha_2}\ln t + U, \quad B = P. \tag{5}$$

Using (5) in (1), we obtain the following reduced system of ODEs:

$$\begin{aligned} &\left(U - \frac{\alpha_4}{\alpha_1}\right)\frac{dR}{d\xi} + R\frac{dU}{d\xi} = 0,\\ &\left(U - \frac{\alpha_4}{\alpha_1}\right)\frac{dU}{d\xi} + \frac{P}{R}\frac{dP}{d\xi} + \frac{k\gamma R^{\gamma-2}}{(1-aR)^{\gamma+1}}\frac{dR}{d\xi} = 0,\\ &\left(U - \frac{\alpha_4}{\alpha_1}\right)\frac{dP}{d\xi} + P\frac{dU}{d\xi} = 0, \end{aligned} \tag{6}$$

which can be solved numerically.

Case B: $\alpha_1 \neq 0$ and $\alpha_2 = 0$.

For this case we obtained the similarity and dependent variables as follows:

$$\xi = x - \left(\frac{\alpha_4}{2\alpha_1}t^2 + \frac{\alpha_3}{\alpha_1}t\right), \quad \rho = R, \quad u = \frac{\alpha_4}{\alpha_1}t + U, \quad B = P. \tag{7}$$

Substituting the variables from (7) in (1) we obtained

$$\begin{aligned} &\left(U - \frac{\alpha_3}{\alpha_1}\right)\frac{dR}{d\xi} + R\frac{dU}{d\xi} = 0,\\ &\left(U - \frac{\alpha_3}{\alpha_1}\right)\frac{dU}{d\xi} + \frac{P}{R}\frac{dP}{d\xi} + \frac{k\gamma R^{\gamma-2}}{(1-aR)^{\gamma+1}}\frac{dR}{d\xi} = 0,\\ &\left(U - \frac{\alpha_3}{\alpha_1}\right)\frac{dP}{d\xi} + P\frac{dU}{d\xi} = 0. \end{aligned} \tag{8}$$

The system of ODEs (8) can be solved numerically.

Case C: $\alpha_1 = 0$ and $\alpha_2 = 0$.

This case yields the similarity and dependent variables as follows:

$$\xi = t, \quad \rho = R, \quad u = \frac{\alpha_4 x}{\alpha_3 + \alpha_4 t} + U, \quad B = P, \tag{9}$$

which reduces (1) to system of ODEs as

$$\begin{aligned} &\frac{dR}{d\xi} + \frac{\alpha_4}{\alpha_3 + \alpha_4\xi}R = 0,\\ &\frac{dU}{d\xi} + \frac{\alpha_4}{\alpha_3 + \alpha_4\xi}U = 0,\\ &\frac{dP}{d\xi} + \frac{\alpha_4}{\alpha_3 + \alpha_4\xi}P = 0. \end{aligned} \tag{10}$$

Further, we obtain the solution of (10) as

$$R = \frac{C_1}{\alpha_3 + \alpha_4\xi}, \quad U = \frac{C_2}{\alpha_3 + \alpha_4\xi}, \quad P = \frac{C_3}{\alpha_3 + \alpha_4\xi}, \tag{11}$$

where C_1, C_2 and C_3 are arbitrary integration constants. The corresponding solution of (1) is

$$\rho = \frac{C_1}{\alpha_3 + \alpha_4 t}, \quad u = \frac{\alpha_4 x + C_2}{\alpha_3 + \alpha_4 t}, \quad P = \frac{C_3}{\alpha_3 + \alpha_4 t}. \tag{12}$$

3. Evolution of weak discontinuities

The matrix form of the governing hyperbolic system is

$$W_t + HW_x = 0, \tag{13}$$

where $W = (\rho, u, B)^T$ is a column vector with superscript T denoting transposition, while H is a matrix with elements $H_{11} = H_{22} = H_{33} = u, H_{12} = \rho, H_{21} = \frac{c^2}{\rho}, H_{13} = H_{31} = 0, H_{23} = \frac{B}{\rho}, H_{32} = B$, where $c^2 = \frac{k\gamma\rho^{\gamma-1}}{(1-a\rho)^{\gamma+1}}$. The matrix H has the eigenvalues